16E Application of matrices to simultaneous equations

When solving equations containing one unknown, only one equation is needed. The equation is transposed to find the value of the unknown. In the case where an equation contains two unknowns, two equations are required to solve the unknowns. These equations are known as simultaneous equations. You may recall the algebraic methods of substitution and elimination used in previous years to solve simultaneous equations.

Matrices may also be used to solve linear simultaneous equations. The following technique demonstrates how to use matrices to solve simultaneous equations involving two unknowns.

Consider a pair of simultaneous equations in the form:

$ax + by = e$

$cx + dy = f$

The equations can be expressed as a matrix equation in the form $AX = B$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the *coefficient matrix*, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$.

Notes

1. A is the matrix of the coefficients of x and y in the simultaneous equations.
2. X is the matrix of the pronumerals used in the simultaneous equations.
3. B is the matrix of the numbers on the right-hand side of the simultaneous equations.

As we have seen from the previous exercise, an equation in the form $AX = B$ can be solved by pre-multiplying both sides by A^{-1}.

$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

WORKED EXAMPLE 13

Solve the two simultaneous linear equations below by matrix methods.

$$2x + 3y = 13$$
$$5x + 2y = 16$$

eBookplus

Tutorial

int-0514

Worked example 13

THINK	WRITE
1 Write the simultaneous equations as a matrix equation in the form $AX = B$. Matrix A is the matrix of the coefficients of x and y in the simultaneous equations, X is the matrix of the pronumerals and B is the matrix of the numbers on the right-hand side of the simultaneous equations.	$AX = B$ $\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$
2 Matrix X is found by pre-multiplying both sides by A^{-1}.	$X = A^{-1}B$
3 Calculate the inverse of A.	$\det A = 2 \times 2 - 3 \times 5$ $= -11$ The inverse (A^{-1}) is $-\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}$
4 Solve the matrix equation by calculating the product of A^{-1} and B and simplify.	$X = A^{-1}B$ $= -\frac{1}{11}\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 16 \end{bmatrix}$ $= -\frac{1}{11}\begin{bmatrix} -22 \\ -33 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
5 Equate the two matrices and solve for x and y.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
6 Write the answers.	The solution to the simultaneous equations is $x = 2$ and $y = 3$.

Simultaneous equations are not just limited to two equations and two unknowns. It is possible to have equations with three or more unknowns. To solve for these unknowns, one equation for each unknown is needed.

Simultaneous equations involving more than two unknowns can be converted to matrix equations in a similar manner to the methods described previously. However, a CAS calculator will be used to find the value of the pronumerals.

Let us consider an ancient Chinese problem that dates back to one of the oldest Chinese mathematics books, *The Nine Chapters on the Mathematical Art.*

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

This information can be converted to equations, using the pronumerals x, y and z to represent the three types of corn, as follows:

$$3x + 2y + 1z = 39$$
$$2x + 3y + 1z = 34$$
$$1x + 2y + 3z = 26$$

(Note the importance of lining up the pronumerals on the left side and the numbers on the right side.)

As was the case earlier with two simultaneous equations, this system of equations can also be written as a matrix equation in the form $AX = B$ as follows:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}$$

X can be solved by pre-multiplying both sides of the equation by A^{-1}. As the order of A is greater than (2 × 2), a CAS calculator should be used to find the inverse (A^{-1}). Try to solve this problem for yourself after reading the following worked example.

WORKED EXAMPLE 14

Use a CAS calculator and matrix methods to solve the following system of equations.

$$x - 2y + z = -2$$
$$-2x + 3y = -3$$
$$2x - z = 4$$

THINK

WRITE/DISPLAY

1 Use the information from the equations to construct a matrix equation. Insert a 0 in the coefficient matrix where the pronumeral is 'missing'.

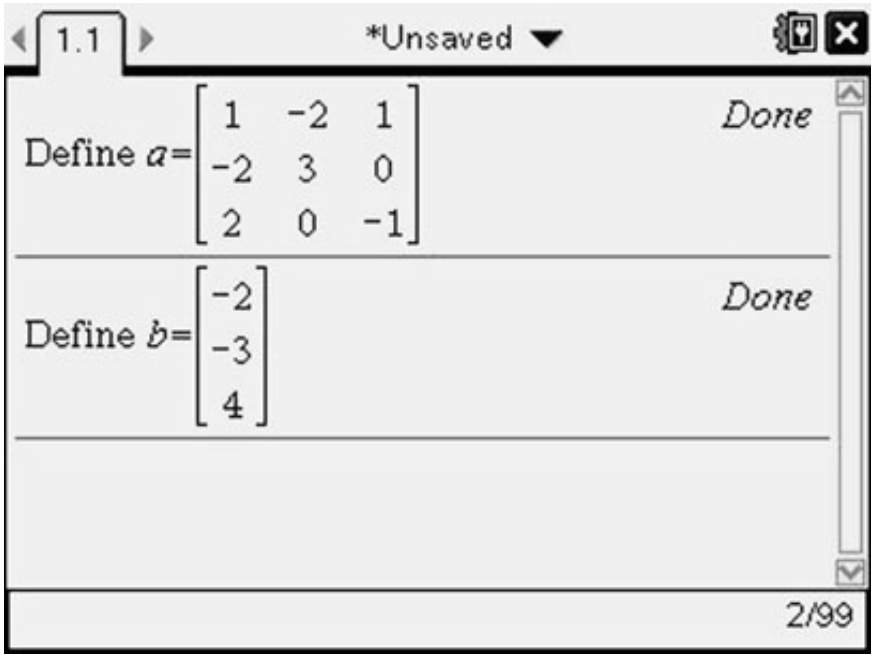
$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

2 Open a Calculator page and complete the entry lines as:

$$\text{Define } a = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\text{Define } b = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Press ENTER after each entry.



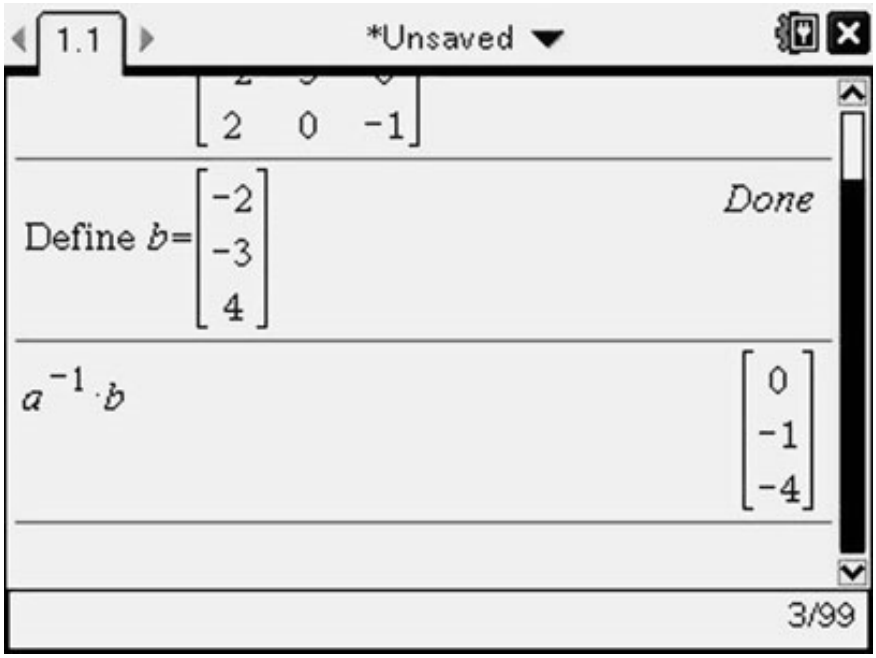
3

X is found by pre-multiplying both sides of the equation by A^{-1} (and hence isolating X on the left and leaving $A^{-1}\ B$ on the right).

Complete the entry line as:

$a^{-1} \times b$

Then press ENTER.



4 Interpret the results and answer the question. You can double-check your answer by substituting these values into the original equations.

The values of the pronumerals are $x = 0$, $y = -1$ and $z = -4$.

Matrix mathematics is a very efficient tool for solving problems with two or more unknowns. As a result, it is used in many areas such as engineering, computer graphics and economics. Matrices may also be applied to solving problems from other modules of the Further Mathematics course, such as break-even analysis, finding the first term and the common difference in arithmetic sequences and linear programming.

When answering problems of this type, take care to follow these steps:

1. Read the problem several times to ensure you fully understand it.
2. Identify the unknowns and assign suitable pronumerals. (Remember that the number of equations needed is the same as the number of unknowns.)
3. Identify statements that define the equations and write the equations using the chosen pronumerals.
4. Use the matrix methods to solve the equations. (Remember, for matrices of order 3 × 3 and higher, use a CAS calculator.)

WORKED EXAMPLE 15

A bakery produces two types of bread, wholemeal and rye. The respective processing times for each batch on the dough-making machine are 12 minutes and 15 minutes, while the oven baking times are 16 minutes and 12 minutes respectively. How many batches of each type of bread should be processed in an 8-hour shift so that both the dough-making machine and the oven are fully occupied?

THINK	WRITE
1 Identify the unknowns and choose a suitable pronumeral for each unknown.	We need to determine the number of batches of wholemeal bread and the number of batches of rye bread. Let x = the number of batches of wholemeal bread Let y = the number of batches of rye bread
2 Write two algebraic equations from the given statements. All times must be expressed in the same units. (8 hours = 480 minutes)	$12x + 15y = 480$ $16x + 12y = 480$
3 Write the simultaneous equations as a matrix equation in the form $AX = B$.	$AX = B$ $\begin{bmatrix} 12 & 15 \\ 16 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 480 \\ 480 \end{bmatrix}$
4 Solve the matrix equation to find the values for x and y. (Alternatively, use a CAS calculator.)	$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{96} \begin{bmatrix} 12 & -15 \\ -16 & 12 \end{bmatrix} \begin{bmatrix} 480 \\ 480 \end{bmatrix}$ $= \frac{-1}{96} \begin{bmatrix} -1440 \\ -1920 \end{bmatrix}$ $= \begin{bmatrix} 15 \\ 20 \end{bmatrix}$
5 Write your answer, relating the pronumerals to the original problem.	$x = 15$ and $y = 20$. To fully utilise the dough-making machine and the oven during an 8-hour shift, 15 batches of wholemeal bread and 20 batches of rye bread should be processed.

REMEMBER

1. A pair of simultaneous equations containing two unknowns in the form:

$$ax + by = e$$
$$cx + dy = f$$

can be expressed as a matrix equation in the form $AX = B$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the coefficient matrix, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$.

2. A CAS calculator should be used to solve simultaneous equations in matrix form when the order is 3 × 3 or greater.
3. When answering problems of this type, take care to follow these steps:

   (a) read the problem several times to ensure you fully understand it

   (b) identify the unknowns and assign suitable pronumerals. (Remember that the number of equations needed is the same as the number of unknowns.)

   (c) identify statements that define the equations and write the equations using the chosen pronumerals. Align the pronumerals.

   (d) use the matrix methods to solve the equations.

EXERCISE 16E Application of matrices to simultaneous equations

1 Solve each of the following matrix equations.

a $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}$

c $\begin{bmatrix} 3 & 7 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -21 \\ 15 \end{bmatrix}$

d $\begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 21 \end{bmatrix}$

2 WE13 Solve the following simultaneous linear equations by matrix methods.

a $-3x + 7y = 65$
$-9x + 6y = -15$

b $3x + 2y = 9$
$6x + 4y = 22$

c $-x + 2y = 0$
$-6x + 14y = 2$

d $4x - y = -3$
$3x - y = -1$

e $x + y = 12$
$x + 4y = 36$

f $6x + 2y = 30$
$2x - y = 10$

3 Use a CAS calculator to solve the following matrix equations.

a $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ -3 \\ 7 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 20 \end{bmatrix}$

c $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$

d $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 15 \\ 6 \end{bmatrix}$

EXAM TIP

These types of question require accurate data entry into the calculator and then correct multiplication of the inverse of the square matrix by the column matrix. Errors in typing the matrix elements into the calculator are common. If an error in data entry has been made … the student scores zero.

[Assessment report 2 2007]

4 WE14 Use a CAS calculator and matrix methods to solve the following system of equations.

$$\begin{aligned} 2x + y + 4z &= 17 \\ 3x - y &= -3 \\ x + 4y + 5z &= 7 \end{aligned}$$

5 MC For the system of simultaneous equations

$$\begin{aligned} x - 2z &= 1 \\ x + y - 3z &= 2 \\ 3x - 2y + z &= 6 \end{aligned}$$

a the coefficient matrix is:

A $\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -3 \\ 3 & -2 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix}$

C $\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -3 \\ -3 & -2 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix}$

E $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix}$

b the solution matrix is:

A $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

B $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

C $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

D $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

E there is no solution.

6 Consider the following two pairs of simultaneous linear equations.

i $6x + 2y = 4$

$9x + 3y = 14$

ii $\frac{1}{2}x - y = 3$

$\frac{3}{2}x - 3y = 9$

a Write each pair of simultaneous equations as a matrix equation in the form *AX - B.*

b Calculate the determinant for both coefficient matrices.

c Find the solution for each pair of simultaneous equations. What do you notice? Suggest a reason for this.

d Transpose the two equations in **i** into $y = mx + c$ form and graph them both on a CAS calculator. How do these graphs relate to your answer from part **C**?

e Transpose the two equations in **ii** into $y = mx + c$ form and graph them both on a CAS calculator. How do these graphs relate to your answer from part **C**?

7 Consider the pair of simultaneous equations.

$$y = 2x + 3$$
$$y = x + 1$$

a Transpose the equations so that they are in the form $ax + by = c$.

b Write the simultaneous equations as a matrix equation in the form $AX = B$.

c Solve the matrix equation, writing the solution in coordinate form.

8 Solve the following set of simultaneous equations using matrix methods on a CAS calculator.

$$2a - 3b + 6c + 2d = 16$$
$$2b + 4c - d = {}^{-}3$$
$${}^{-}a - b - c - d = {}^{-}4.5$$
$$0.1a + 0.4b - 0.6c + 1.2d = 3.1$$

9 Consider the following problem studied by the Babylonians. (*Note:* We have substituted square metres, instead of square yards, as the units of area.)

There are two fields whose total area is 1800 square metres. One produces grain at the rate of $\frac{2}{3}$ of a bushel per square metre while the other produces grain at the rate of $\frac{1}{2}$ a bushel per square metre. If the total yield is 1100 bushels, what is the size of each field?

Use matrix methods to solve the problem.

10 The sum of two numbers is 79 and their difference is 25. Find the two numbers by setting up two linear simultaneous equations and solving them using matrix methods.

11 An arithmetic sequence has the fifth term equal to 13.5 and the twelfth term equal to 31. To find the first term, a, of this sequence and the common difference, d, the following two equations can be used:

$$a + 4d = 13.5$$
$$a + 11d = 31$$

Use matrix methods to find the first term and the common difference for this arithmetic sequence.

12 WE15 At a car spray-painting company, each car receives two coats of paint, which have to be completed within one day. There are two types of cars that this company spray paints — sedans and utilities.

The times are displayed in the following table.

	Stage of painting	
	1st coat	**2nd coat**
Sedan	5 minutes	9 minutes
Utility	7 minutes	8 minutes
Total time available for each stage	140 minutes	183 minutes

To fully utilise the company's time, how many sedans and utilities should be planned for in a day?

13 MC In an alternative to the scoring for Australian Rules, a team gains g points for a goal and b points for a behind. In a recent match, Geelong obtained 69 points for scoring 7 goals and 3 behinds and Colling wood obtained 113 points for scoring 11 goals and 7 behinds.

a Which of the following matrix equations describes the scoring in this game?

A $\begin{bmatrix} 7 & 3 \\ 11 & 7 \end{bmatrix} \begin{bmatrix} b \\ g \end{bmatrix} = \begin{bmatrix} 69 \\ 113 \end{bmatrix}$

B $\begin{bmatrix} 7 & 3 \\ 11 & 7 \end{bmatrix} \begin{bmatrix} g \\ b \end{bmatrix} = \begin{bmatrix} 113 \\ 69 \end{bmatrix}$

C $\begin{bmatrix} 7 & 3 \\ 11 & 7 \end{bmatrix} \begin{bmatrix} b \\ g \end{bmatrix} = \begin{bmatrix} 113 \\ 69 \end{bmatrix}$

D $\begin{bmatrix} 11 & 7 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} g \\ b \end{bmatrix} = \begin{bmatrix} 113 \\ 69 \end{bmatrix}$

E None of these

b The number of points awarded for each goal is:

A 6 points

B 7 points

C 8 points

D 9 points

E 10 points

14 The cost (in dollars) of manufacturing electronic components, *d,* is related to the number of components produced, *n,* by the formula $d = 6000 + 2.5n$. The revenue, *d* (in dollars), generated from selling *n* components is given by the formula $d = 4.5n - 8000$. Use matrix methods to calculate the number of components that need to be manufactured so that the manufacturing cost and revenue are equal.

15 The table below displays the attendance numbers and the box-office takings for the first three shows of a new stage play.

Show	Adults	Children	Pensioners	Box-office takings ($)
First	40	20	5	945
Second	50	15	15	1165
Third	30	0	40	800

Use matrix methods and a CAS calculator to calculate the ticket prices for adults, children and pensioners.

16 Use matrix methods to find two numbers, where twice a number plus three times another number is 166 and the sum of the two numbers is 58.

17 A factory produces two different models of transistor radios. Each model requires two workers to assemble it. The time taken by each worker varies according to the following table.

	Worker 1	Worker 2
Model A	5 minutes	5 minutes
Model B	18 minutes	4 minutes
Maximum time available for each worker	360 minutes	150 minutes

a Use matrix methods to calculate how many of each model should be produced so that each worker is used for the total time available.

b If the company makes $2.50 on each model A sold and $4.00 on each model B sold, what is the maximum amount of revenue from the sales?

18 The sum of the first 15 terms in an arithmetic sequence is 633 and the 30th term is 187.4. To find the first term, *a,* of this sequence and the common difference, *d,* the following two equations can be used

$$15a + 105d = 633$$
$$a + 29d = 187.4$$

Use matrix methods to find the first term and the common difference for this arithmetic sequence.

eBookplus

Digital doc

WorkSHEET 16.2