Diplomarbeit[1] *on*

"The asymptotics of the gravitational field and the memory effect"

by Volker Schlue[2]

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I. Basic Constructions

The asymptotics of the gravitational field at future null infinity shall be described for a spatially confined source in an otherwise vacuous spacetime.

Let us think of the source, which could be a coalescence of a neutron star binary, to be initially surrounded in space by a spacelike surface S_0, diffeomorphic to the sphere S^2, lying in a maximal spacelike hypersurface Σ_0 of vanishing linear momentum (see figure 1).

Σ_0 itself is contained in the spacetime manifold $\mathcal{M}$, endowed with the Lorentzian metric g, so that at each point $x \in S_0$, there exist future directed *null normals* $\underline{L}_x$ and L_x,

$$(T_xS_0)^\perp \cap N_x = \langle \underline{L}_x \rangle \cup \langle L_x \rangle \,,$$

which are unique up to a positive real factor. (Here $(T_xS_0)^\perp$ denotes the orthogonal complement to T_xS_0 in $T_x\mathcal{M}$, and N_x stands for the null cone in

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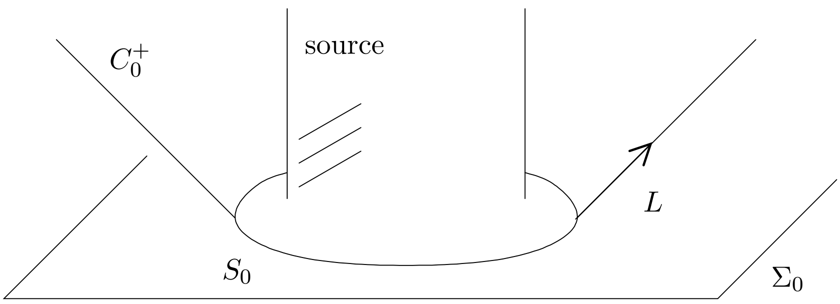

Figure 1: The source and the basic surfaces in spacetime.

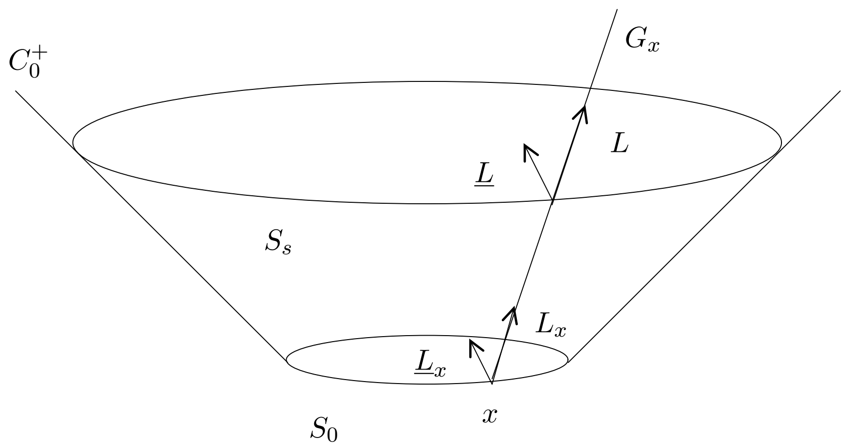

Figure 2: The sections S_s of the null hypersurface C_0^+.

$T_x\mathcal{M}$, while $\langle V \rangle$ is the linear span of a vector V.) One of the two generates a null line which projects to the interior of S_0 on Σ_0; we refer to this, which we take to be $\underline{L}_x$, as the *incoming*, while to the other, L_x, as the *outgoing* null normal.

Making a continuous choice of the outgoing null normal at each point we obtain a vectorfield L on S_0, which is unique up to a positive multiplicative function on S_0, satisfying

$$g(L, L) = 0\,, \quad g(L, X) = 0 \quad : \text{on } S_0, \tag{1}$$

where X is a vectorfield tangential to S_0. The incoming null normals form a vectorfield on S_0, once subject to the normalization condition

$$g(L, \underline{L}) = -2 \quad : \text{on } S_0. \tag{2}$$

We refer to $\underline{L}$ as the *conjugate* null vectorfield to L, satisfying

$$g(\underline{L}, \underline{L}) = 0\,, \quad g(\underline{L}, X) = 0 \quad : \text{on } S_0, \tag{3}$$

with X as above.

Consider the null geodesics G_x initiating at $x \in S_0$ with tangent vector L_x (see figure 2), then

$$C_0^+ = \bigcup_{x \in S_0} G_x$$

is a *null hypersurface*. We make the **basic assumption** that S_0 is chosen such that the generators G_x of C_0^+ have no future end points.

Definition. C is called a null hypersurface if at each point $q \in C$ the induced metric $g_q|_{T_qC}$ is degenerate.

L is extended to C_0^+ to be the vectorfield, coinciding with L on S_0, whose integral curves are the G_x; then in particular

$$\nabla_L L = 0\,. \tag{4}$$

We define the *affine function* s on C_0^+ by:

$$s|_{S_0} = 0\,, \quad L \cdot s = 1\,.$$

The level sets of s, S_s, are sections[3] of C_0^+ (see figure 2). $\underline{L}$ is extended to C_0^+ to be the conjugate vectorfield to L on each S_s. We shall think of the surfaces S_s as *wave fronts.*

In the following we define several quantities on the surfaces S_s describing the geometric properties of each, and their change along the generators of the null hypersurfaces. Their initial values on S_0 are assumed to be given, while their propagation must be in accordance with the *Einstein vacuum equations*[4]

$$S_{\mu\nu} = 0\,; \tag{5}$$

(here $S_{\mu\nu}$ are the components of the Ricci curvature of spacetime in an arbitrary frame).

Definition 1. The *induced metric* on S_s is

$$\gamma_x = g|_{T_x S_s}\,, \quad (x \in S_s).$$

A vectorfield X on S_0 can be extended to a *Jacobi field* along the generators of C_0^+ according to

$$[L, X] = 0\,; \tag{6}$$

along a generator, X can then be thought of as the displacement to a nearby generator.

Definition 2. The *null second fundamental form* of S_s is

$$\chi(X, Y) = g(\nabla_X L, Y)\,, \quad (X, Y \in \mathcal{X}(S_s)).$$

This is a bilinear, symmetric form, for[5]

$$g(\nabla_X L, Y) - g(\nabla_Y L, X) = -g(L, \nabla_X Y) + g(L, \nabla_Y X) = -g(L, [X, Y]) \stackrel{(1)}{=} 0\,.$$

[3]One may view C_0^+ as a line bundle over S_0.

[4]In fact, we consider a spacetime for which they hold at any point, so that the vacuum equations can be viewed as an algebraic property of the curvature tensor, referred to as the *trace condition.*

[5]The connection is metric, and symmetric.

Now, for any Jacobi fields X,Y, as above,

$$\begin{aligned}\frac{\partial}{\partial s}\gamma(X,Y) &= L\cdot g(X,Y)\\ &= g(\nabla_L X,Y)+g(X,\nabla_L Y)\overset{(6)}{=}g(\nabla_X L,Y)+g(X,\nabla_Y L)\\ &= 2\,\chi(X,Y)\,, \qquad (7)\end{aligned}$$

which is the **first variational formula** (for sections of C_0^+).

Definition 3. The *torsion* of S_s is

$$\zeta(X)=\frac{1}{2}\,g(\nabla_X L,\underline{L})\,,\quad (X\in\mathcal{X}(S_s)).$$

This is a 1-form on S_s.

Let $(E_A : A=1,2)$ be an orthonormal frame for S_0, extended along the generators of C_0^+ according to

$$\nabla_L E_A=-\zeta_A L\,, \qquad (8)$$

where $\zeta_A=\zeta(E_A)$. Then $(E_A : A=1,2)$ is an orthonormal frame for S_s, since

$$L\cdot\gamma(E_A,E_B)\overset{(8),(1)}{=}0\,,$$

while

$$\gamma(E_A,E_B)=\delta_{AB}\quad : \text{on } S_0,$$

and further

$$L\cdot g(L,E_A)\overset{(4)}{=}-\zeta_A g(L,L)=0$$

while

$$g(L,E_A)=0\quad : \text{on } S_0,$$

and

$$\begin{aligned}L\cdot g(\underline{L},E_A) &\overset{(8)}{=} g(\nabla_L\underline{L},E_A)-\zeta_A g(\underline{L},L)\\ &\overset{(2)}{=} -2\,\zeta_A+2\,\zeta_A=0\end{aligned}$$

while

$$g(\underline{L},E_A)\overset{(3)}{=}0\quad : \text{on } S_0.$$

In the last step we have used the following Lemma, whose easy proof is omitted here.

Lemma. *Let Z be the S_s-tangent vectorfield corresponding to the 1-form ζ, then*

$$\nabla_L\underline{L}=-2Z\,.$$

Since any Jacobi field X is tangent to the surfaces S_s, we can expand[6] X in the frame $(E_A : A = 1, 2)$,

$$X = X^A E_A.$$

Then

$$\nabla_L X = \frac{\mathrm{d}X^A}{\mathrm{d}s} E_A - X^A \zeta_A L\,. \tag{9}$$

So on one hand

$$g(\nabla_L X, E_B) = \frac{\mathrm{d}X^B}{\mathrm{d}s}\,,$$

while on the other

$$g(\nabla_L X, E_B) \overset{(6)}{=} \chi(X, E_B) = \chi_{AB} X^A\,,$$

where $\chi_{AB} = \chi(E_A, E_B)$; we conclude that

$$\frac{\mathrm{d}X^A}{\mathrm{d}s} = \chi_{AB} X^B\,. \tag{10}$$

We are now heading towards the **propagation equations for $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$**, in which the curvature components α and β appear.

Definition 4. The *null decomposition* of the curvature tensor of spacetime relative to the null frame $(E_A : A = 1, 2; \underline{L}, L)$ consists of the symmetric tensors α, $\underline{\alpha}$, on S_s,

$$\alpha_{AB} = R(E_A, L, E_B, L) \qquad \underline{\alpha}_{AB} = R(E_A, \underline{L}, E_B, \underline{L})$$

the 1-forms β, $\underline{\beta}$, on S_s,

$$\beta_A = \frac{1}{2} R(E_A, L, \underline{L}, L) \qquad \underline{\beta}_A = \frac{1}{2} R(E_A, \underline{L}, \underline{L}, L)$$

and the scalars ρ, σ,

$$-\rho\, \epsilon_{AB} \epsilon_{CD} = R(E_A, E_B, E_C, E_D) \qquad \sigma\, \epsilon_{AB} = \frac{1}{2} R(E_A, E_B, \underline{L}, L)$$

where ϵ is the volume form of S_s.

Remark 1. The decomposition is complete only under the condition (5). Also, by virtue of (5), α and $\underline{\alpha}$ are *traceless*, as discussed in Remark 2.

[6] We use the summation convention.

Recall, for any vectorfields V, W, Z, on $\mathcal{M}$,

$$[\nabla_V, \nabla_W]Z - \nabla_{[V,W]}Z = R(V,W) \cdot Z\,, \tag{11}$$

so in particular,

$$\nabla_L^2 X \overset{(6)}{=} \nabla_L \nabla_X L \overset{(4)}{=} R(L,X) \cdot L\,,$$

a vectorfield tangential to C_0^+, for

$$g(L, R(L,X) \cdot L) = R(L,L,L,X) = 0\,,$$

which may be expanded in terms of $(E_A : A = 1,2; L)$. For any such vectorfield Y,

$$Y = Y^A E_A + Y^L L\,,$$

where

$$Y^A = g(E_A, Y)\,, \quad Y^L = -\frac{1}{2} g(\underline{L}, Y)\,,$$

so

$$\begin{aligned} R(L,X) \cdot L &= \sum_{A=1}^{2} R(E_A, L, L, X) E_A - \frac{1}{2} R(\underline{L}, L, L, X) L \\ &= -\sum_{A=1}^{2} \alpha_{AB} X^B E_A + \beta_B X^B L\,. \end{aligned}$$

Remark. The curvature tensor

$$R(W,Z,X,Y) = g(W, R(X,Y) \cdot Z)$$

is antisymmetric in (W,Z) as well as (X,Y). Moreover it has the pair symmetry

$$R(W,Z,X,Y) = R(X,Y,W,Z).$$

On the other hand,

$$\nabla_L^2 X \overset{(9)}{=} \frac{\mathrm{d}^2 X^A}{\mathrm{d}s^2} E_A - 2 \frac{\mathrm{d}X^A}{\mathrm{d}s} \zeta_A L - X^A \frac{\mathrm{d}\zeta_A}{\mathrm{d}s} L\,,$$

so that we conclude, (see also (10)),

$$\frac{\mathrm{d}^2 X^A}{\mathrm{d}s^2} = -\alpha_{AB} X^B \tag{12}$$

$$\frac{\mathrm{d}\zeta_A}{\mathrm{d}s} + 2 \sum_{B=1}^{2} \chi_{AB} \zeta_B = -\beta_A\,. \tag{13}$$

In view of (12), (10), (linear homogeneous second order system of ordinary differential equations), we can write

$$X^A = M^A{}_B X^B_{(0)}\,,$$

where M is the so called deformation matrix. Then

$$\frac{\mathrm{d}M^A{}_B}{\mathrm{d}s} = \chi_{AC} M^C{}_B$$

$$\frac{\mathrm{d}^2 M^A{}_B}{\mathrm{d}s^2} = -\alpha_{AC} M^C{}_B\,,$$

from which we see that, in terms of matrices,

$$\begin{aligned}\frac{\mathrm{d}\chi}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\right] \\ &= \frac{\mathrm{d}^2M}{\mathrm{d}s^2}M^{-1} - \frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1} \\ &= -\chi^2 - \alpha\,,\end{aligned}$$

or

$$\frac{\mathrm{d}\chi_{AB}}{\mathrm{d}s} = -\sum_{C=1}^{2} \chi_{AC}\chi_{CB} - \alpha_{AB}\,. \tag{14}$$

Remark 2. In any null frame $(E_A : A = 1, 2; \underline{L}, L)$,

$$g^{-1} = \sum_{A=1}^{2} E_A \otimes E_A - \frac{1}{2}\underline{L} \otimes L - \frac{1}{2} L \otimes \underline{L}\,, \tag{15}$$

thus

$$\begin{aligned}S(L,L) &= \sum_{A=1}^{2} R(E_A, L, E_A, L) - \frac{1}{2}R(\underline{L}, L, L, L) - \frac{1}{2}R(L, L, \underline{L}, L) \\ &= \sum_{A=1}^{2} \alpha_{AA} = \mathrm{tr}\,\alpha\,,\end{aligned}$$

so by virtue of (5),

$$\mathrm{tr}\,\alpha = 0\,.$$

Similarly for $\underline{\alpha}$.

The **propagation equation for tr $\boldsymbol{\chi}$** will therefore contain no curvature terms, in fact

$$\frac{\mathrm{d}\,\mathrm{tr}\,\chi}{\mathrm{d}s} = -|\chi|^2\,, \tag{16}$$

where

$$|\chi|^2 = \sum_{A,B=1}^{2} (\chi_{AB})^2 = \mathrm{tr}\,(\chi^2)\,.$$

Finally we consider the trace-free parts (denoted by a caret), and express the above equations in a Jacobi frame $(X_a : a = 1, 2)$,

$$X_a = X_a^A E_A\,.$$

The formula (7) simply becomes

$$\frac{\partial \gamma_{ab}}{\partial s} = 2\chi_{ab}\,, \tag{17}$$

where $\chi_{ab} = \chi(X_a, X_b)$. For (14) we note

$$\chi_{ab} = X_a^A X_b^B \chi_{AB}\,,$$

and readily calculate, using (10),

$$\frac{\partial \chi_{ab}}{\partial s} = \sum_{A=1}^{2} \chi_{aA}\chi_{Ab} - \alpha_{ab}\,,$$

or, since

$$(\gamma^{-1})^{cd} = \sum_{A=1}^{2} E_A^c E_A^d\,, \tag{18}$$

where E_A^a are the components of E_A in the frame $(X_a : a = 1, 2)$,

$$\frac{\partial \chi_{ab}}{\partial s} - \chi_a{}^c \chi_{cb} = -\alpha_{ab}\,, \tag{19}$$

where $\chi_a{}^b = (\gamma^{-1})^{bc}\chi_{ac}$. Setting

$$\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2}\gamma_{ab}\,\mathrm{tr}\,\chi\,, \tag{20}$$

we have

$$|\chi|^2 = |\hat{\chi}|^2 + \frac{1}{2}(\mathrm{tr}\,\chi)^2\,,$$

so that (16) may equally well be written as

$$\frac{\partial \mathrm{tr}\,\chi}{\partial s} + \frac{1}{2}(\mathrm{tr}\,\chi)^2 = -|\hat{\chi}|^2\,. \tag{21}$$

Moreover, using (19), (17), and (21),

$$\frac{\partial \hat{\chi}_{ab}}{\partial s} = \hat{\chi}_a{}^c \hat{\chi}_{cb} + \frac{1}{2}\gamma_{ab}|\hat{\chi}|^2 - \alpha_{ab}\,, \tag{22}$$

and

$$\mathrm{tr}\,\frac{\partial \hat{\chi}}{\partial s} = 2|\hat{\chi}|^2\,,$$

yielding

$$\frac{\hat{\partial} \hat{\chi}_{ab}}{\partial s} = \hat{\chi}_a{}^c \hat{\chi}_{cd} - \frac{1}{2}\gamma_{ab}|\hat{\chi}|^2 - \alpha_{ab}\,;$$

but $\hat{\chi}$ is symmetric and trace-free, so in fact

$$\frac{\hat{\partial} \hat{\chi}_{ab}}{\partial s} = -\alpha_{ab}\,. \tag{23}$$

Regarding (13) one proceeds as from (14) to (19) to obtain

$$\frac{\partial \zeta_a}{\partial s} + \chi_a{}^b \zeta_b = -\beta_a \,, \tag{24}$$

where $\beta_a = \beta(X_a)$.

However, the propagation equations are coupled to *elliptic systems* on each section S_s, a consequence of the embedding of the spacelike surfaces in a vacuum spacetime. The **Codazzi equations** for a section S_s of C_0^+ are

$$\begin{aligned}(\nabla\!\!\!/_X \chi)(Y,Z) &- (\nabla\!\!\!/_Y \chi)(X,Z) \\ &- \chi(X,Z)\zeta(Y) + \chi(Y,Z)\zeta(X) = R(Z,L,X,Y)\,,\end{aligned} \tag{25}$$

for any vectorfields X, Y, Z tangential to S_s. (Here $\nabla\!\!\!/$ denotes the covariant derivative corresponding to γ.)

Remark 3. As indicated before, in a null frame $(E_1, E_2; E_3 = \underline{L}, E_4 = L)$,

$$S(E_B, L) = \sum_{A=1}^{2} R(E_A, E_B, E_A, L) - \frac{1}{2} R(\underline{L}, E_B, L, L) - \frac{1}{2} R(L, E_B, \underline{L}, L)\,,$$

so by virtue of (5),

$$\beta_B = -\sum_{A=1}^{2} R_{ABA4}\,. \tag{26}$$

Therefore in the trace of (25) no curvature components other than β appear, in fact,

$$\sum_{A=1}^{2} \nabla\!\!\!/_A \chi_{BA} - \nabla\!\!\!/_B \mathrm{tr}\,\chi + \sum_{A=1}^{2} \zeta_A \chi_{BA} - \zeta_B \mathrm{tr}\,\chi = -\beta_B\,. \tag{27}$$

Introducing the *divergence operator* intrinsic to the surfaces S_s,

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\theta_a = \nabla\!\!\!/^b \theta_{ab} = (\gamma^{-1})^{bc} \nabla\!\!\!/_c \theta_{ab}\,,$$

where θ is a symmetric, 2-covariant tensorfield on S_s, it holds

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\chi_a - \nabla\!\!\!/_a \mathrm{tr}\,\chi + \chi_a{}^b \zeta_b - \mathrm{tr}\,\chi\,\zeta_a = -\beta_a\,, \tag{28}$$

expressing (27) in a Jacobi frame; note that

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\chi_a = \nabla\!\!\!/^b \chi_{ab} \overset{(18)}{=} \sum_{A=1}^{2} \nabla\!\!\!/_A \chi_{BA} X_a^B\,,$$

$$\nabla\!\!\!/_a \mathrm{tr}\,\chi = \nabla\!\!\!/_B \mathrm{tr}\,\chi\, X_a^B\,.$$

Using (28) one may eliminate the curvature term in (24) to obtain

$$\frac{\partial \zeta_a}{\partial s} + \mathrm{tr}\,\chi\,\zeta_a = \mathrm{d}\!\!\!/\mathrm{iv}\,\chi_a - \nabla\!\!\!/_a \mathrm{tr}\,\chi\,.$$

Substituting for χ from (20) in (28),

$$\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi}_a + \hat{\chi}_a{}^b\zeta_b = \frac{1}{2}\left(\nabla\!\!\!/_a \mathrm{tr}\,\chi + \mathrm{tr}\,\chi\,\zeta_a\right) - \beta_a\,, \tag{29}$$

we arrive at the *null Codazzi equation*:

$$\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi} + \hat{\chi}\cdot\zeta = \frac{1}{2}\left(\nabla\!\!\!/\,\mathrm{tr}\,\chi + \mathrm{tr}\,\chi\,\zeta\right) - \beta\,. \tag{30}$$

Definition 5. The *conjugate null second fundamental form* of S_s is

$$\underline{\chi}(X,Y) = g(\nabla_X \underline{L}, Y)\,, \quad (X, Y \in \mathcal{X}(S_s))\,.$$

This is a bilinear, symmetric form, like χ.

The antisymmetric part of the propagation equation for $\underline{\chi}$,

$$0 = L\cdot\underline{\chi}(X,Y) - L\cdot\underline{\chi}(Y,X)\,,$$

whose calculation shall not distract us here, is

$$\mathrm{c}\!\!/\mathrm{url}\,\zeta = \sigma - \frac{1}{2}\hat{\chi}\wedge\hat{\underline{\chi}}\,, \tag{31}$$

where

$$\mathrm{c}\!\!/\mathrm{url}\,\zeta = \frac{1}{2}\epsilon^{ab}(\nabla\!\!\!/_a\zeta_b - \nabla\!\!\!/_b\zeta_a)\,,$$

$$\hat{\chi}\wedge\hat{\underline{\chi}} = \frac{1}{2}\epsilon^{ab}(\hat{\chi}_a{}^c\hat{\underline{\chi}}_{cb} - \hat{\chi}_b{}^c\hat{\underline{\chi}}_{ca})\,;$$

ζ enters the equations as a consequence of the Lemma on page 4.

The **Gauss equations** for a section S_s of C_0^+ in spacetime are

$$\begin{aligned} &R\!\!\!/(W,Z,X,Y) \\ &\qquad + \chi(W,X)\underline{\chi}(Z,Y) - \chi(W,Y)\underline{\chi}(Z,X) = R(W,Z,X,Y)\,, \end{aligned} \tag{32}$$

where W, X, Y, Z are vectorfields tangential to S_s; (here $R\!\!\!/$ denotes the curvature tensor of S_s). Since S_s is a 2-dimensional Riemannian manifold,

$$R\!\!\!/_{DCAB} = K\left(\delta_{DA}\delta_{CB} - \delta_{DB}\delta_{CA}\right),$$

(where K is the Gauss curvature of S_s), and by definition,

$$R_{DCAB} = -\rho\,\epsilon_{DC}\epsilon_{AB}\,.$$

Therefore, setting $W = X = E_A$, $Z = Y = E_B$ in (32), and summing over A and B,

$$K + \frac{1}{2}\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\chi\cdot\underline{\chi} = -\rho$$

where

$$\chi \cdot \underline{\chi} = \operatorname{tr} \chi \underline{\chi} = \sum_{A,B=1}^{2} \chi_{AB} \underline{\chi}_{BA} ,$$

or, in terms of

$$\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} \delta_{AB} \operatorname{tr} \chi , \tag{33}$$

$$K + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}} = -\rho . \tag{34}$$

Definition 6. The *mass aspect function* associated to the surfaces S_s is

$$\mu = K + \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} - \operatorname{d\!\!/iv} \zeta . \tag{35}$$

Remark 4. The origin of this name will be outlined in part II. In particular see (60), and Remark 5.

Substituting for K in (34), we obtain an equation, supplementing (31) to a *Hodge system* for ζ:

$$\operatorname{d\!\!/iv} \zeta = -\mu - \rho + \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}} . \tag{36}$$

This system is seen to be coupled to a propagation equation for μ.

Let B_0 be the interior of S_0 in Σ_0, and B_d, $d > 0$, the set of all points in Σ_0 with distance from B_0 less than d. Let d^* be a distance large compared to the radius of S_0, and

$$C_*^- = \partial J^+(\Sigma_0 \backslash B^*)$$

the boundary of the domain of dependence of $B^* = B_{d^*}$; (here $J^+(B)$ is the causal future of $B \subset \Sigma_0$). Finally, we define

$$S_0^* = C_0^+ \cap C_*^- , \tag{37}$$

a spherical spacelike surface, see figure 3, which we shall take to contain the *event of observation* $p \in S_0^*$ – here the wave front “hits” the observer.

Remark. From the physical point of view the event $p \in \mathcal{M}$ is determined *first*, e.g. by a measurement on earth, and the surface S_0 surrounding the source is then to be chosen in the neighborhood of the intersection of the source with the boundary of the causal past of p. Moreover, d^* is then required to be the smallest distance for which B^* contains the causal past of p in Σ_0,

$$d^* = \inf \left\{ d > 0 \,|\, J^-(p) \cap \Sigma_0 \subset B_d \right\} .$$

The surface S_0^*, as defined above, will then, in general, *not* contain p, rather p lies in a neighborhood of S_0^*; see figure 4. However, compared to d^*, which is gigantic, e.g. of the order of the distance of the earth from the source,

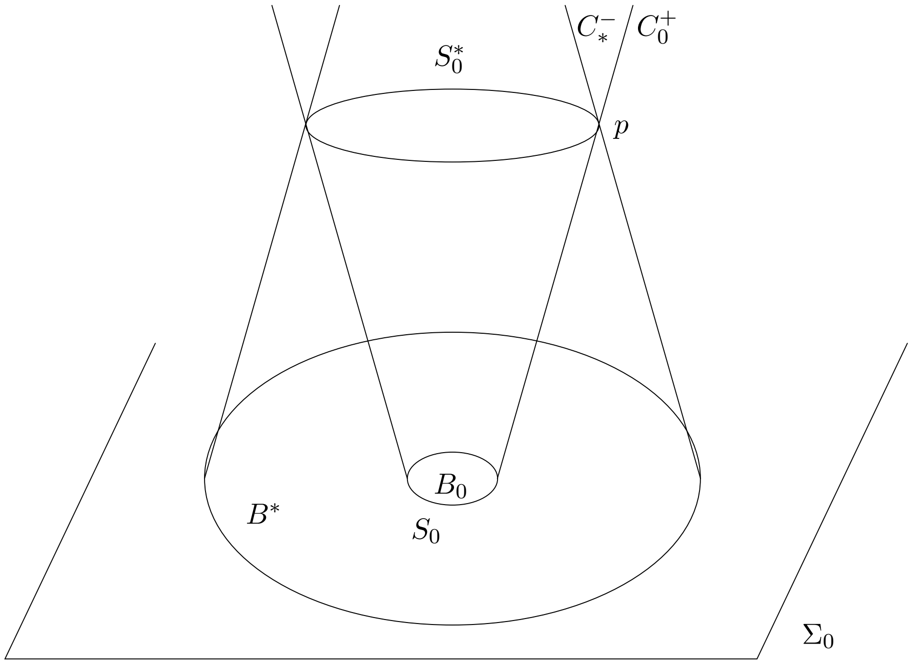

Figure 3: The surface S_0^*.

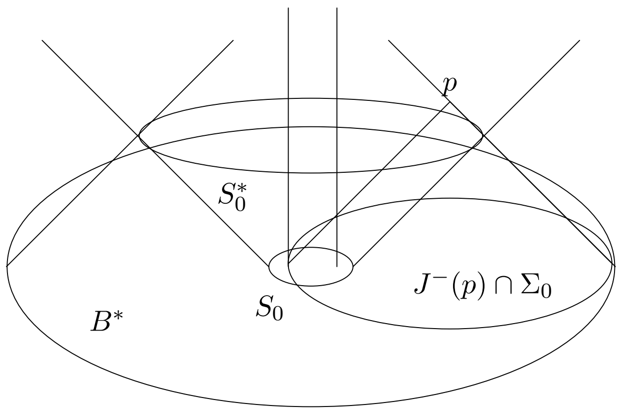

Figure 4: The construction of the surface S_0^* for an event $p \in \mathcal{M}$ determined in advance.

the distance of p from S_0^* can be neglected. Even more important is the recognition that the offset of p from S_0^* is not of physical relevance at all, for there is *no canonical choice* of S_0. The arbitrariness in the choice of S_0 (apart from the stated requirements) is carried over to the wavefronts, which therefore cannot be physically accessible as such. What does have physical relevance, and can be measured, shall concern us is part III.

Recall that the null normal vectorfields L, $\underline{L}$, on S_0, and extended to C_0^+, are only unique up to a positive multiplicative function a on S_0,

$$L \to aL\,, \quad \underline{L} \to \frac{1}{a}\underline{L}\,.$$

In the case that $\mathrm{tr}\,\chi$ and $\mathrm{tr}\,\underline{\chi}$ have opposite signs, we can fix the normalization on S_0^*, and consequently on C_0^+, by

$$\mathrm{tr}\,\chi + \mathrm{tr}\,\underline{\chi} = 0 \quad : \text{on } S_0^*\,. \tag{38}$$

Next, $\underline{L}$, L, are extended to C_*^- in the same way as they were extended from S_0 to C_0^+, just with the roles of L, $\underline{L}$ interchanged. Here, we do not label the level sets of the affine function by its respective value, but rather by the associated *retarded time*:

$$u = 2(r_0^* - r^*)\,; \tag{39}$$

here r_0^*, r^* are the area radii of S_0^* and the level set respectively.

Definition. The *area radius* of a spherical surface S, i.e. diffeomorphic to S^2, with metric γ, satisfies

$$4\pi r^2 = \mathrm{Area}(S) = \int_S \mathrm{d}\mu_\gamma\,. \tag{40}$$

Finally L, $\underline{L}$, are extended to each C_u^+, the inner component of the causal past of S_u^*, similarly, namely: L is defined to be the geodesic vectorfield on C_u^+ whose integral curves are the generators of C_u^+, and which coincides with L just constructed on $S_u^* \subset C_*^-$. On each level set $S_{u,s}$ of the affine function on C_u^+,

$$L \cdot s = 1\,, \quad s|_{S_u^*} = r_u^*\,, \tag{41}$$

which are sections of C_u^+, $\underline{L}$ is defined to be the conjugate null vectorfield to L. Thus the null normal pair $\underline{L}$, L, is constructed in the entire (future) domain of dependence of B^*, and fixed by (38)[7]; see figure 5.

The discussion of the Codazzi equations for sections S_s of C_0^+ can be repeated for sections S_u^* of C_*^- with the roles of L and $\underline{L}$ interchanged. (Note that the signs in the corresponding equations to (25), here for the torsion terms, and (26) are reversed.) This leads to the *conjugate null Codazzi equation*:

$$\mathrm{d}\!\!/\mathrm{iv}\,\hat{\underline{\chi}} - \hat{\underline{\chi}} \cdot \zeta = \frac{1}{2}(\nabla\!\!\!/\,\mathrm{tr}\,\underline{\chi} - \mathrm{tr}\,\underline{\chi}\,\zeta) + \underline{\beta}\,. \tag{42}$$

[7]With the choice (38), $T = \frac{1}{2}(L + \underline{L})$ is the *binormal* of S_0^*.

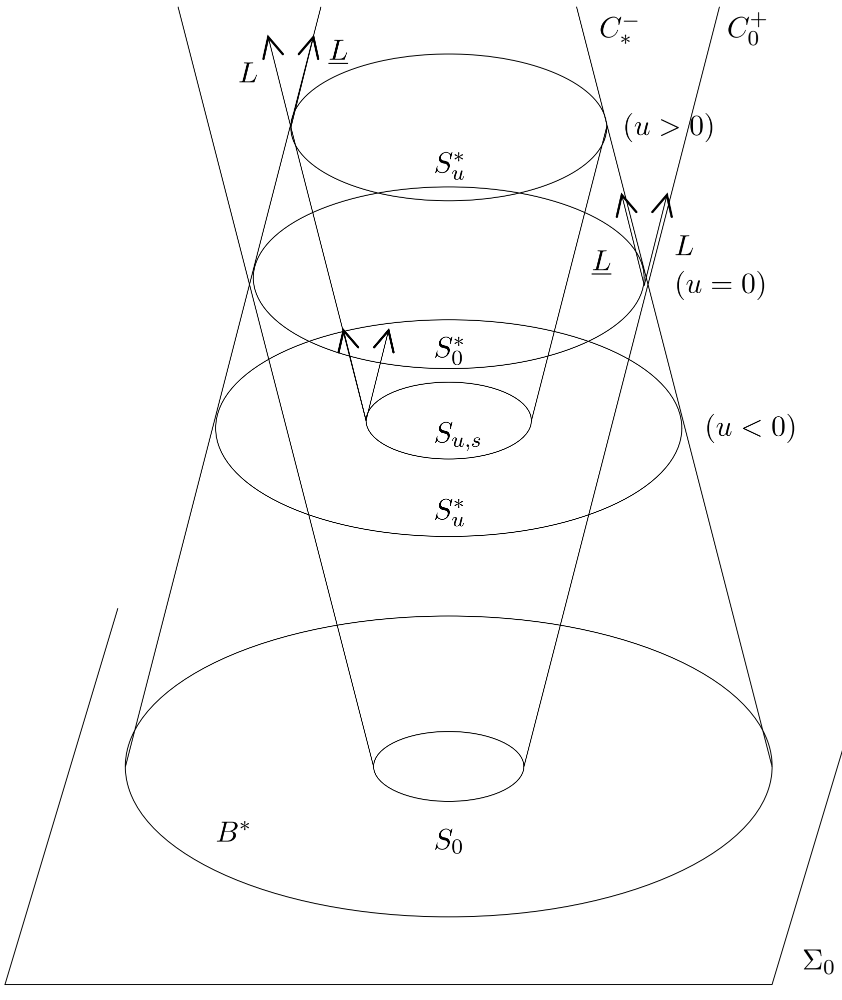

Figure 5: Extension of the null normals to the domain of dependence of B^*.

Definition 7. The *conjugate mass aspect function* associated to the surfaces $S_{u,s}$ is

$$\underline{\mu} = K + \frac{1}{4}\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} + \mathrm{d}\!\!/\mathrm{iv}\,\zeta \tag{43}$$

Then (36) may also be written as

$$\mathrm{d}\!\!/\mathrm{iv}\,\zeta = \underline{\mu} + \rho - \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}}\,. \tag{44}$$

The entire discussion of the gravitational field that follows relies on the fact, that for any experiment under consideration, the radius of S_0, introduced as a sphere surrounding the source, in comparison to the radius of S_0^*, introduced as a surface whose neighborhood in spacetime contains the events of observation, is *negligible*. The mathematical limit which is of interest to us here, is where

$$d^* \longrightarrow \infty\,, \tag{45}$$

which is made more precise in part II, or equivalently where B^* exhausts Σ_0.

II. Asymptotic Behavior

In order to make precise limit statements we have to formalize (45):
Let

$$\Phi_0 : S^2 \to S_0^*$$

be a diffeomorphism, and

$$\Phi_u = \underline{\varphi}_{\underline{s}(u)} \circ \Phi_0 : S^2 \to S_u^*\,,$$

where $\underline{\varphi}_{\underline{s}}$ is the 1-par.-group generated by $\underline{L}$ on C_*^-, and $\underline{s}(u)$ is the value of the affine function corresponding to the retarded time u, that is

$$\underline{\varphi}_{\underline{s}(u)}(S_0^*) = S_u^*\,,$$

then Φ_u is also a diffeomorphism, (fixed by Φ_0). Similarly we define the diffeomorphism

$$\Phi_{u,s} = \varphi_{s-r_u^*} \circ \Phi_u : S^2 \to S_{u,s}\,,$$

where φ_s is the 1-par.-group generated by L on C_u^+; see also (41). The choice of Φ_0 requires that S_0^* is constructed. But in the limiting process (45) such a choice can be made for each value of d^*, or equivalently r_0^*, so in fact there is a whole family of diffeomorphisms $\Phi_0^{(r_0^*)}$, which we fix by the condition that as (45) happens the image of a given point $x \in S^2$, $\Phi_0^{(r_0^*)}(x)$, as a function of r_0^*, traces a generator of C_0^+.

Let w be a $S_{u,s}$-*tangent* p-covariant tensorfield, (w vanishes if any of its entries are L or $\underline{L}$), satisfying

$$|\,w\,|_\gamma = \sqrt{w^{a_1\cdots a_p} w_{a_1\cdots a_p}} = \mathcal{O}(r^{-q})\,;$$

(where $w^{a_1}{}_{a_2\cdots a_p} = (\gamma^{-1})^{a_1 b_1} w_{b_1 a_2\cdots a_p}$). We investigate the limit of

$$\tilde{w}^{(r_0^*)} = \Phi_u{}^*({r_u^*}^{-p+q} w)\,,$$

a p-covariant tensorfield on S^2, as r_0^* tends to infinity; (note that w is here the restriction to S_u^*). In the case of existence, we simply write

$$r^q w \longrightarrow W\,,$$

where W still depends on u.

Lemma. *Let Dw and $\underline{D}w$ denote the projections of $\pounds_L w$ and $\pounds_{\underline{L}} w$ to $S_{u,s}$ respectively, then*

$$\frac{\partial}{\partial s}\Phi_{u,s}{}^* w = \Phi_{u,s}{}^*(Dw)\,, \tag{46}$$

and

$$(\underline{L}\cdot u)\frac{\partial}{\partial u}\Phi_{u,s}{}^* w = \Phi_{u,s}{}^*(\underline{D}w)\,. \tag{47}$$

Proof. By the definition of Lie derivative,

$$\pounds_L w = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t{}^* w|_{t=0}\,,$$

and

$$\Phi_{u,s}{}^*\varphi_t{}^* = (\varphi_t \circ \Phi_{u,s})^* = \Phi_{u,s+t}\,;$$

for the second additionally

$$\frac{\mathrm{d}\underline{s}}{\mathrm{d}u} = (\underline{L}\cdot u)^{-1}\,.$$

Now, as (45) S_u^* moves along C_u^+ and $r_u^* \to \infty$, hence (see also (50))

$$\begin{aligned}
&\Big| W - \Phi_u{}^*({r_u^*}^{-p+q} w)\Big|_{\Phi_u{}^*({r_u^*}^{-2}\gamma)} \\
&\quad \le C\int_{r_u^*}^{\infty}\left|\frac{\partial}{\partial s}\Phi_{u,s}{}^*({r_{u,s}}^{-p+q} w)\right|_{\Phi_{u,s}{}^*(r_{u,s}^{-2}\gamma)} \mathrm{d}s \\
&\quad \overset{(46)}{=} C\int_{r_u^*}^{\infty}\left|D({r_{u,s}}^{-p+q} w)\right|_{r_{u,s}^{-2}\gamma}\mathrm{d}s \\
&\quad = C\int_{r_u^*}^{\infty}({r_{u,s}}^2)^{\frac{p}{2}}\,\Big|\,(L\cdot r_{u,s}^{-p})\,{r_{u,s}}^q\, w + {r_{u,s}}^{-p}\, D({r_{u,s}}^q\, w)\,\Big|_\gamma\,\mathrm{d}s\,,
\end{aligned}$$

but, using the definition (40),

$$L \cdot r_{u,s} = \frac{1}{8\pi\, r_{u,s}} L \cdot \int_{S_{u,s}} \mathrm{d}\mu_\gamma = \\ = \frac{1}{8\pi\, r_{u,s}} \int_{S_{u,s}} \operatorname{tr}\chi \, \mathrm{d}\mu_\gamma = \frac{r_{u,s}}{2} \overline{\operatorname{tr}\chi} \,,$$

for it is

$$L \cdot \sqrt{\det\gamma} = \frac{1}{2} \frac{1}{\sqrt{\det\gamma}} \det\gamma \, (\gamma^{-1})^{ab} \frac{\partial \gamma_{ab}}{\partial s} \overset{(17)}{=} \operatorname{tr}\chi \, \sqrt{\det\gamma} \,,$$

and substituting in favor of r,

$$s(r^*_u) = r^*_u \,, \quad s(\infty) = \infty \,, \quad \frac{\mathrm{d}s}{\mathrm{d}r} = (L \cdot r)^{-1} \,,$$

we get

$$\left| W - {\Phi_u}^* ({r^*_u}^{-p+q} w) \right|_{{\Phi_u}^*({r^*_u}^{-2}\gamma)} \\ \leq C \int_{r^*_u}^{\infty} \left| -\frac{p}{2} \overline{\operatorname{tr}\chi}\, r^q w + \frac{\partial}{\partial s}(r^q w) \right|_\gamma \left(\frac{r}{2} \overline{\operatorname{tr}\chi} \right)^{-1} \mathrm{d}r \tag{48}$$

where it is used that

$$(Dw)_{a_1 \cdots a_p} = (\pounds_L w)(\Pi X_{a_1}, \ldots, \Pi X_{a_p}) = \\ = L \cdot (w(X_{a_1}, \ldots, X_{a_p})) \\ - w([L, X_{a_1}], X_{a_2}, \ldots, X_{a_p}) \\ - \ldots - w(X_{a_1}, \ldots, X_{a_{p-1}}, [L, X_{a_p}]) \\ \overset{(6)}{=} \frac{\partial}{\partial s} w_{a_1 \cdots a_p} \,.$$

Therefore the **existence of W** depends on the integrability of the propagation equation for $r^q w$, more precisely we must have

$$\left| -\frac{p}{2} \overline{\operatorname{tr}\chi}\, r^q w + \frac{\partial}{\partial s}(r^q w) \right|_\gamma = \mathcal{O}(r^{-1-\varepsilon}) \,, \tag{49}$$

for some $\varepsilon > 0$, (see also (52)).

Proposition 1. *For each fixed u,*

$$\gamma \longrightarrow \overset{\circ}{\gamma} \,, \tag{50}$$

where $\overset{\circ}{\gamma}$ is independent of u, and isometric to the standard metric on S^2.

While the existence of the limit relies on results for χ, the independence of u, namely

$$\frac{\partial \overset{\circ}{\gamma}}{\partial u} = 0\,, \tag{51}$$

relies on those for $\underline{\chi}$. Here, and in the following, I only prove limits related to the χ system, introduced in part I, given by (16), and (29), and whose analysis can be found in [Ch-II].

Proof of (50): Here the expression in question, (49), simply reads

$$\begin{aligned}
&\left| -\overline{\mathrm{tr}\,\chi}\,\gamma + \frac{\partial}{\partial s}\gamma \right|_{\gamma} = \\
&\quad = |\, 2\hat{\chi} + \gamma\,\mathrm{tr}\,\chi - \gamma\,\overline{\mathrm{tr}\,\chi}\,|_{\gamma} \leq 2\,|\,\hat{\chi}\,|_{\gamma} + 2\left|\,\mathrm{tr}\,\chi - \mathrm{tr}\,\underline{\chi}\,\right|_{\gamma} \\
&\quad = \mathcal{O}(r^{-2})
\end{aligned}$$

since both

$$|\,\hat{\chi}\,|_{\gamma} = \mathcal{O}(r^{-2})\,, \quad \text{and } |\,\mathrm{tr}\,\chi - \overline{\mathrm{tr}\,\chi}\,|_{\gamma} = \mathcal{O}(r^{-2})\,.$$

Q.E.D.

The metric $\overset{\circ}{\gamma}$ can be considered to be the standard metric on S^2, since it can be shown that it is of Gauss curvature 1.

Proposition 2. *For each fixed u,*

$$r\,\mathrm{tr}\,\chi \longrightarrow 2\,, \tag{52}$$

$$r\,\mathrm{tr}\,\underline{\chi} \longrightarrow -2\,, \tag{53}$$

and furthermore, setting

$$g = r^2(\mathrm{tr}\,\chi - \overline{\mathrm{tr}\,\chi})\,, \tag{54}$$

for each value of u,

$$g \longrightarrow H\,, \tag{55}$$

where H is a function on S^2, which is independent of u:

$$\frac{\partial H}{\partial u} = 0\,. \tag{56}$$

Proof of (55): This relies on the propagation equations

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{r}{2}\overline{\mathrm{tr}\,\chi}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{tr}\,\chi = -\frac{1}{2}(\mathrm{tr}\,\chi)^2 - |\hat{\chi}|^2\,,$$

which are already known, and that for the mean value,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\overline{\operatorname{tr}\chi} &= -\overline{\operatorname{tr}\chi}^2 + \frac{1}{2}\overline{(\operatorname{tr}\chi)^2} - \overline{|\hat{\chi}|^2} \\ &= -\frac{1}{2}\overline{\operatorname{tr}\chi}^2 + \frac{1}{2}\overline{(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi})^2} - \overline{|\hat{\chi}|^2}\,,\end{aligned}$$

for then

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}s} &= r^2\overline{\operatorname{tr}\chi}(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) - \frac{1}{2}r^2((\operatorname{tr}\chi)^2 - (\overline{\operatorname{tr}\chi})^2) \\ &\quad - \frac{1}{2}r^2\overline{(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi})^2} - r^2(|\hat{\chi}|^2 - \overline{|\hat{\chi}|^2}) \\ &= -\frac{1}{2}(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi})g - \frac{1}{2}r^{-2}\overline{g^2} - r^2(|\hat{\chi}|^2 - \overline{|\hat{\chi}|^2}) \\ &= -\frac{1}{2}r^{-2}(g^2 + \overline{g^2}) - r^2(|\hat{\chi}|^2 - \overline{|\hat{\chi}|^2}) \\ &= \mathcal{O}(r^{-2})\,,\end{aligned}$$

since $\quad g = r^2(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) = \mathcal{O}(1)\,, \quad$ and $|\hat{\chi}| = \mathcal{O}(r^{-2})\,.$

Q.E.D.

The following two limits turn out to be directly related to experiment, (see (147) and (148)).

Proposition 3. *For each fixed u,*

$$r^2\,\hat{\chi} \longrightarrow \Sigma\,, \tag{57}$$

$$r\,\underline{\hat{\chi}} \longrightarrow \Xi\,, \tag{58}$$

where Σ and Ξ are 2-covariant symmetric tensorfields on S^2, which are trace-free relative to $\overset{\circ}{\gamma}$.

Proof of (57): The quantity in question, (49), reads in this case

$$\left| -\overline{\operatorname{tr}\chi}\, r^2\hat{\chi} + \frac{\partial}{\partial s}(r^2\hat{\chi})\right|_\gamma = \left| r^2\frac{\partial}{\partial s}\hat{\chi}\right|_\gamma .$$

Recalling that $\hat{\chi}$ is trace-free and symmetric, we rewrite (22) as in the deduction of (23) as

$$\frac{\partial\hat{\chi}_{ab}}{\partial s} = \gamma_{ab}|\hat{\chi}|^2 - \alpha_{ab}\,.$$

Then

$$\left| r^2\frac{\partial}{\partial s}\hat{\chi}\right|_\gamma \leq 2r^2\,|\,\hat{\chi}\,|_\gamma^2 + r^2\,|\,\alpha\,|_\gamma = \mathcal{O}(r^{-\frac{3}{2}})\,,$$

since

$$|\,\hat{\chi}\,|_\gamma = \mathcal{O}(r^{-2})\,, \quad |\,\alpha\,|_\gamma = \mathcal{O}(r^{-\frac{7}{2}})\,,$$

as stated below, (see (73)). Q.E.D.

The next limit relies on a result for the ζ system, introduced in part I and given by (24), (31) and (36), but not further treated here, (see [Ch-II]).

Proposition 4. *For each fixed value of u,*

$$r^2\zeta \longrightarrow Z\,, \tag{59}$$

where Z is a 1-form on S^2.

Proof of (59): Here (49) reads

$$\left|-\frac{1}{2}\,\overline{\mathrm{tr}\,\chi}\,r^2\zeta+\frac{\partial}{\partial s}(r^2\zeta)\right|_\gamma=\left|\frac{1}{2}\,\overline{\mathrm{tr}\,\chi}\,r^2\zeta+r^2\frac{\partial}{\partial s}\zeta\right|_\gamma\,.$$

Introducing (20) into the propagation equation (24), we conclude that the above is bounded by

$$\frac{1}{2}|\mathrm{tr}\,\chi-\overline{\mathrm{tr}\,\chi}|\left|r^2\zeta\right|_\gamma+\left|r^2\hat{\chi}\right|_\gamma|\,\zeta\,|_\gamma+r^2\,|\,\beta\,|_\gamma=\mathcal{O}(r^{-\frac{3}{2}})\,,$$

since

$$|\,\zeta\,|_\gamma=\mathcal{O}(r^{-2})\,,\quad |\,\beta\,|_\gamma=\mathcal{O}(r^{-\frac{7}{2}})\,,$$

(see (75) below). Q.E.D.

In view of the Gauss-Bonnet formula, let us integrate the mass aspect functions (35) and (43) over any of the surfaces $S_{u,s}$:

$$\begin{aligned}\int_S\mu\,\mathrm{d}\mu_\gamma&=\int_S\underline{\mu}\,\mathrm{d}\mu_\gamma=\\&=4\pi+\frac{1}{4}\int_S\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi}\,\mathrm{d}\mu_\gamma=\frac{8\pi m}{r}\,,\end{aligned}\tag{60}$$

where m is the *Hawking mass.*

Definition 8. The *Hawking mass* of the surface $S_{u,s}$ is

$$m_{u,s}=\frac{r_{u,s}}{2}\left(1+\frac{1}{16\pi}\int_{S_{u,s}}\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi}\,\mathrm{d}\mu_\gamma\right)\,. \tag{61}$$

Remark 5. Consider the surface S_0^*. Since the null normal pair $(\underline{L},L)$ is here chosen to satisfy (38), it follows that

$$\mathrm{tr}\,k=0\quad :\text{on }S_0^*\,, \tag{62}$$

where for the present

$$k(X,Y)=g(\nabla_X T,Y)\,,\quad (X,Y\in\mathcal{X}(S_0^*))\,, \tag{63}$$

and

$$T=\frac{1}{2}(L+\underline{L}) \tag{64}$$

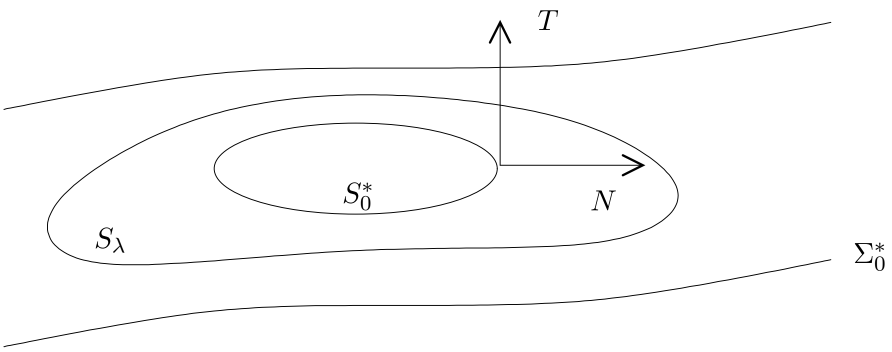

Figure 6: The family of surfaces S_λ in Σ_0^*.

is the unit future directed timelike vectorfield on S_0^*. Let us assume that S_0^* lies in a maximal spacelike hypersurface Σ_0^* (see figure 6),then

$$\operatorname{tr} k = 0 \quad : \text{on } \Sigma_0^* \,, \tag{65}$$

where k is the *second fundamental form* of Σ_0^*,

$$k(X,Y) = g(\nabla_X T, Y)\,, \quad (X, Y \in \mathcal{X}(\Sigma_0^*))\,, \tag{66}$$

and T is the unit future-directed timelike normal to Σ_0^*, in particular

$$g(T,X) = 0\,, \quad (X \in \mathcal{X}(\Sigma_0^*))\,,$$

thus (62) extends to (65). Furthermore let us assume that not only (65) holds, but that the second fundamental form vanishes identically,

$$k_{ij} = 0 \quad : \text{on } \Sigma_0^* \,,$$

then the "first" *constraint equation*, namely the (T,T) component of the Einstein equations,

$$\operatorname{tr} \overline{S} + (\operatorname{tr} k)^2 - |k|^2 = 4\,T(T,T)\,,$$

reduces to

$$\operatorname{tr} \overline{S} = 0\,;$$

(here $\overline{S}$ denotes the intrinsic Ricci curvature of Σ_0^*). Under the additional hypotheses that Σ_0^* is Euclidean at infinity, and that there exists a coordinate system (x_1, x_2, x_3) in the neighborhood of infinity in Σ_0^* relative to which the metric components take the form[8]

$$\overline{g}_{ij} = \left(1 + \frac{M}{2\rho}\right)^4 \delta_{ij} + \mathcal{O}_2(\rho^{-1-\varepsilon})\,, \quad (\varepsilon > 0)\,, \tag{67}$$

[8]It is said $f = \mathcal{O}_m(\rho^{-k})$ if $\partial^l f = \mathcal{O}(r^{-k-l})$ for all $l = 0, \ldots, m$, where $\partial^l f$ stands for all partial derivatives of order l relative to (x_1, x_2, x_3).

where $\rho = \sqrt{\sum_{i=1}^3 x_i^2}$, it can straight be proven (see [Ch-I]) that for a suitable family of surfaces S_λ, whose interiors exhaust Σ_0^*, the Hawking mass of S_λ tends to the *total mass* M,

$$m_\lambda \longrightarrow M\,; \tag{68}$$

note that (62), or (38), also implies that

$$\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} = -(\mathrm{tr}\,\theta)^2\,,$$

once we set

$$N = \frac{1}{2}(L - \underline{L})\,,$$

the unit normal to S_0^* relative to Σ_0^*,

$$g(T, N) = 0\,,$$

and

$$\theta(X,Y) = \overline{g}(\overline{\nabla}_X N, Y)\,, \quad (X, Y \in \mathcal{X}(S_0^*))\,,$$

(where the overbar denotes the restriction to Σ_0^*), so m_λ in (68) is not (61) rather

$$m_\lambda = \frac{r_\lambda}{2}\left(1 - \frac{1}{16\pi}\int_{S_\lambda}(\mathrm{tr}\,\theta)^2\,\mathrm{d}\mu_\gamma\right)\,.$$

The reason for the interpretation of M in (68) as mass, and its identification with *total energy* is the fact that the calculation of the *ADM-energy*[9] of Σ_0^*,

$$E = \frac{1}{4}\lim_{\rho\to\infty}\int_{S_\rho}\sum_{i,j}\left(\partial_i \overline{g}_{ij} - \partial_j \overline{g}_{ii}\right)\mathrm{d}S_j\,,$$

(where S_ρ is the coordinate sphere of radius ρ), with $\overline{g}_{ij}$ given by (67) yields

$$E = 4\pi\, M.$$

Without proof, we note the following important limits.

Proposition 5. *For each fixed u,*

$$r^3\mu \longrightarrow N\,, \tag{69}$$

$$r^3\underline{\mu} \longrightarrow \underline{N}\,, \tag{70}$$

where N and $\underline{N}$ are functions on S^2.

[9] It is referred to the definition of the notion of total energy as introduced by Arnowitt, Deser and Misner.

As a consequence of (60),

$$m = \frac{1}{2}\frac{1}{4\pi r^2}\int_S r^3 \mu \, \mathrm{d}\mu_\gamma \, ,$$

we have

$$m \longrightarrow M \, , \tag{71}$$

the *Bondi mass*:

$$\begin{aligned} M &= \frac{1}{2}\overline{N} = \frac{1}{2}\underline{\overline{N}} \\ &= \frac{1}{8\pi}\int_{S^2} N \, \mathrm{d}\mu_{\overset{\circ}{\gamma}} \, . \end{aligned} \tag{72}$$

Some of the above results depend directly on the decay of the curvature components, but all of them do so indirectly since the analysis of the χ and ζ system assumes bounds for the spacetime curvature to obtain estimates for the quantities controlling the geometry of C_0^+, (see [Ch-II]). These bounds have to be seen as a consequence of the *Main Theorem* in [ChKl-Stability].

Proposition 6. *The components of the null decomposition (Definition 4) of the curvature have the following asymptotic properties:*

$$|\,\alpha\,|_\gamma = \mathcal{O}(r^{-\frac{7}{2}}) \tag{73}$$

$$|\,\underline{\alpha}\,|_\gamma = \mathcal{O}(r^{-1}) \tag{74}$$

$$|\,\beta\,|_\gamma = \mathcal{O}(r^{-\frac{7}{2}}) \tag{75}$$

$$\left|\,\underline{\beta}\,\right|_\gamma = \mathcal{O}(r^{-2}) \tag{76}$$

$$|\,\rho\,|_\gamma = \mathcal{O}(r^{-3}) \tag{77}$$

$$|\,\sigma\,|_\gamma = \mathcal{O}(r^{-3}) \, . \tag{78}$$

Four of these quantities in fact have limits:

Proposition 7. *For each fixed u,*

$$r\,\underline{\alpha} \longrightarrow A \, , \tag{79}$$

where A is a 2-covariant symmetric tensorfield on S^2 which is trace-free relative to $\overset{\circ}{\gamma}$,

$$r^2\,\underline{\beta} \longrightarrow B \, , \tag{80}$$

where B is a 1-form on S^2,

$$r^3\,\rho \longrightarrow P \, , \tag{81}$$

$$r^3\,\sigma \longrightarrow Q \, , \tag{82}$$

where P and Q are functions on S^2.

Let us recall the **Bianchi identity**:

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0\,, \quad (X,Y,Z \in \mathcal{X}(\mathcal{M}))\,.$$

In components, relative to an arbitrary frame $(E_\mu : \mu = 0,\ldots,3)$, where

$$(\nabla_{E_\nu} R)(E_\alpha, E_\mu)\cdot E_\lambda = \nabla_\nu R^\beta{}_{\lambda\alpha\mu} E_\beta\,,$$

it reads

$$\nabla_\nu R^\beta{}_{\lambda\alpha\mu} + \nabla_\alpha R^\beta{}_{\lambda\mu\nu} + \nabla_\mu R^\beta{}_{\lambda\nu\alpha} = 0\,.$$

Setting $\alpha = \beta$ and sum, we obtain

$$\nabla_\alpha R^\alpha{}_{\lambda\mu\nu} = \nabla_\mu S_{\lambda\nu} - \nabla_\nu S_{\lambda\mu}\,,$$

(recall here that $S_{\mu\nu} = \sum_\alpha (R_{\alpha\nu})^\alpha{}_\mu$). However, by virtue of (5) the r.h.s. vanishes identically, so in fact the *first contracted Bianchi identities in a vacuum spacetime* are

$$\nabla_\alpha R^\alpha{}_{\lambda\mu\nu} = 0\,. \tag{83}$$

Each of the equations controlling the components of the null decomposition and allowing to prove the limits of Proposition 7 are certain components of (83). More explicitly (83) reads

$$(g^{-1})^{\alpha\beta} g((\nabla_{E_\alpha} R)(E_\mu, E_\nu)\cdot E_\lambda, E_\beta) = 0\,, \tag{84}$$

which is written out in a null frame according to (15).

Proof of (79): I choose $E_\mu = \underline{L}$, $E_\nu = E_B$, $E_\lambda = E_A$ in (84), then

$$\begin{aligned} 0 \;=\; & \sum_{C=1}^{2} g((\nabla_{E_C} R)(\underline{L}, E_B)\cdot E_A, E_C) \\ & - \frac{1}{2} g((\nabla_{\underline{L}} R)(\underline{L}, E_B)\cdot E_A, L) \\ & - \frac{1}{2} g((\nabla_L R)(\underline{L}, E_B)\cdot E_A, \underline{L})\,. \end{aligned} \tag{85}$$

The derivatives of the curvature are passed on the null frame by Leibniz rule:

$$\begin{aligned} (\nabla_{E_C} R)(\underline{L}, E_B)\cdot E_A &= \nabla_{E_C}(R(\underline{L}, E_B)\cdot E_A) \\ &\quad - R(\nabla_{E_C}\underline{L}, E_B)\cdot E_A - R(\underline{L}, \nabla_{E_C} E_B)\cdot E_A - R(\underline{L}, E_B)\cdot \nabla_{E_C} E_A \\[2ex] (\nabla_{\underline{L}} R)(\underline{L}, E_B)\cdot E_A &= \nabla_{\underline{L}}(R(\underline{L}, E_B)\cdot E_A) \\ &\quad - R(\nabla_{\underline{L}}\underline{L}, E_B)\cdot E_A - R(\underline{L}, \nabla_{\underline{L}} E_B)\cdot E_A - R(\underline{L}, E_B)\cdot \nabla_{\underline{L}} E_A \\[2ex] (\nabla_L R)(\underline{L}, E_B)\cdot E_A &= \nabla_L(R(\underline{L}, E_B)\cdot E_A) \\ &\quad - R(\nabla_L\underline{L}, E_B)\cdot E_A - R(\underline{L}, \nabla_L E_B)\cdot E_A - R(\underline{L}, E_B)\cdot \nabla_L E_A \end{aligned} \tag{86}$$

So the following vectorfields have to be expanded in a null frame:

$$\begin{array}{lll} \nabla_{E_B} E_A\,, & \nabla_{E_A} \underline{L}\,, & \\ \nabla_{\underline{L}} E_A\,, & \nabla_{\underline{L}} \underline{L}\,, & \nabla_{\underline{L}} L\,, \\ \nabla_{L} E_A\,, & \nabla_{L} \underline{L}\,. & \end{array}$$

Since in general

$$X = \sum_{A=1}^{2} g(X, E_A) E_A - \frac{1}{2} g(X, L)\underline{L} - \frac{1}{2} g(X, \underline{L}) L\,, \quad (X \in \mathcal{X}(\mathcal{M}))\,,$$

the following coefficients will appear in the expansions:

$$\chi_{AB} = g(\nabla_{E_A} L, E_B)\,, \quad \underline{\chi}_{AB} = g(\nabla_{E_A} \underline{L}, E_B)\,,$$

$$\zeta_A = \frac{1}{2}\, g(\nabla_{E_A} L, \underline{L})\,,$$

$$\underline{\xi}_A = \frac{1}{2}\, g(\nabla_{\underline{L}} \underline{L}, E_A)\,, \quad \omega = -\frac{1}{4}\, g(\nabla_{\underline{L}} \underline{L}, L)\,.$$

Now:

$$\begin{aligned} \nabla_{E_B} E_A &= \nabla\!\!\!/_{E_B} E_A + \frac{1}{2}\, \chi_{AB}\, \underline{L} + \frac{1}{2}\, \underline{\chi}_{AB}\, L \\ \nabla_{E_A} \underline{L} &= \sum_{B=1}^{2} \underline{\chi}_{AB} E_B + \zeta_A\, \underline{L} \\ \nabla_{\underline{L}} E_A &= \Pi \nabla_{\underline{L}} E_A + \zeta_A\, \underline{L} + \underline{\xi}_A\, L \\ \nabla_{\underline{L}} \underline{L} &= \sum_{A=1}^{2} 2\, \underline{\xi}_A E_A + 2\, \omega\, \underline{L} \\ \nabla_{\underline{L}} L &= \sum_{A=1}^{2} 2\, \zeta_A E_A - 2\, \omega\, L \\ \nabla_{L} E_A &\overset{(8)}{=} -\zeta_A\, L \\ \nabla_{L} \underline{L} &= -\sum_{A=1}^{2} 2\, \zeta_A E_A \end{aligned} \tag{87}$$

Substituting (86) in (85) and using the formulas (87) I proceed termwise:

$$\begin{aligned} &+ \sum_{C=1}^{2} g(\nabla_{E_C}(R(\underline{L}, E_B) \cdot E_A), E_C) = \\ &= \sum_{C=1}^{2} E_C \cdot g(R(\underline{L}, E_B) \cdot E_A, E_C) - \sum_{C=1}^{2} g(R(\underline{L}, E_B) \cdot E_A, \nabla_{E_C} E_C) \\ &= \sum_{C=1}^{2} E_C \cdot \epsilon_{CA}{}^{*}\underline{\beta}_B - \sum_{C=1}^{2} R(\nabla\!\!\!/_{E_C} E_C, E_A, \underline{L}, E_B) \end{aligned}$$

$$-\frac{1}{2}\,\mathrm{tr}\,\chi\,\underline{\alpha}_{AB}+\frac{1}{2}\,\mathrm{tr}\,\underline{\chi}\,\rho\,\delta_{AB}+\frac{1}{2}\,\mathrm{tr}\,\underline{\chi}\,\sigma\,\epsilon_{AB} \tag{88}$$

$$-\sum_{C=1}^{2} g(R(\nabla_{E_C}\underline{L},E_B)\cdot E_A,E_C)=\rho\,\mathrm{tr}\,\underline{\chi}\,\delta_{AB}-\rho\,\underline{\chi}_{AB}-{}^*\zeta_A{}^*\underline{\beta}_B \tag{89}$$

$$\begin{aligned}-\sum_{C=1}^{2} g(R(\underline{L},\nabla_{E_C}E_B)\cdot E_A,E_C)&=\\ =\quad -\sum_{C=1}^{2} R(E_C,E_A,\underline{L},\nabla\!\!\!/_{E_C}E_B)&-\sum_{C=1}^{2}\underline{\chi}_{BC}\epsilon_{CA}\,\sigma\end{aligned} \tag{90}$$

$$\begin{aligned}-\sum_{C=1}^{2} g(R(\underline{L},E_B)\cdot\nabla_{E_C}E_A,E_C)&=\\ =\quad -\sum_{C=1}^{2} R(E_C,\nabla\!\!\!/_{E_C}E_A,\underline{L},E_B)&\\ +\frac{1}{2}\sum_{C=1}^{2}\chi_{AC}\,\underline{\alpha}_{CB}-\frac{1}{2}\,\rho\,\underline{\chi}_{AB}+\frac{1}{2}\,\sigma\,\sum_{C=1}^{2}\underline{\chi}_{AC}\,\epsilon_{BC}&\end{aligned} \tag{91}$$

$$\begin{aligned}-\frac{1}{2}\,g(\nabla_{\underline{L}}(R(\underline{L},E_B)\cdot E_A),L)&=\\ =\quad -\frac{1}{2}\underline{L}\cdot g(R(\underline{L},E_B)\cdot E_A,L)&+\frac{1}{2}g(R(\underline{L},E_B)\cdot E_A,\nabla_{\underline{L}}L)\\ =\quad \frac{1}{2}\,\delta_{AB}\,\underline{L}\cdot\rho+\frac{1}{2}\,\epsilon_{AB}\,\underline{L}\cdot\sigma&+{}^*\zeta_A{}^*\underline{\beta}_B+\omega\,\rho\,\delta_{AB}+\omega\,\sigma\,\epsilon_{AB}\end{aligned} \tag{92}$$

$$+\frac{1}{2}\,g(R(\nabla_{\underline{L}}\underline{L},E_B)\cdot E_A,L)=-{}^*\underline{\xi}_B{}^*\beta_A-\omega\,\rho\,\delta_{AB}-\omega\,\sigma\,\epsilon_{AB} \tag{93}$$

$$+\frac{1}{2}\,g(R(\underline{L},\nabla_{\underline{L}}E_B)\cdot E_A,L)=\frac{1}{2}R(L,E_A,\underline{L},\Pi\nabla_{\underline{L}}E_B)-\underline{\xi}_B\,\beta_A \tag{94}$$

$$+\frac{1}{2}\,g(R(\underline{L},E_B)\cdot\nabla_{\underline{L}}E_A,L)=\frac{1}{2}R(L,\Pi\nabla_{\underline{L}}E_A,\underline{L},E_B)+\zeta_A\,\underline{\beta}_B \tag{95}$$

$$\begin{aligned}-\frac{1}{2}\,g(\nabla_L(R(\underline{L},E_B)\cdot E_A),\underline{L})&=\\ =\quad -\frac{1}{2}L\cdot g(R(\underline{L},E_B)\cdot E_A,\underline{L})&+\frac{1}{2}g(R(\underline{L},E_B)\cdot E_A,\nabla_L\underline{L})\\ =\quad -\frac{1}{2}\,L\cdot\underline{\alpha}_{AB}-{}^*\zeta_A{}^*\underline{\beta}_B&\end{aligned} \tag{96}$$

$$+\frac{1}{2}\,g(R(\nabla_L\underline{L},E_B)\cdot E_A,\underline{L})=-{}^*\zeta_B{}^*\underline{\beta}_A \tag{97}$$

$$+\frac{1}{2}\,g(R(\underline{L},\nabla_L E_B)\cdot E_A,\underline{L}) = +\zeta_B\,\underline{\beta}_A \tag{98}$$

$$+\frac{1}{2}\,g(R(\underline{L},E_B)\cdot\nabla_L E_A,\underline{L}) = +\zeta_A\,\underline{\beta}_B \tag{99}$$

Here

$$^*\underline{\beta}_A = \sum_{B=1}^{2}\underline{\beta}_B\,\epsilon_{BA} \qquad ^*\zeta_A = \sum_{B=1}^{2}\zeta_B\,\epsilon_{BA}$$

are the components of the dual 1-forms of $\underline{\beta}$ and ζ respectively.
First consider the terms yielding intrinsic derivatives of the curvature:

$$\begin{aligned}
&\sum_{C=1}^{2}E_C\cdot\epsilon_{CA}{}^*\underline{\beta}_B - \sum_{C=1}^{2}R(\nabla\!\!\!/_{E_C}E_C,E_A,\underline{L},E_B)\\
&\qquad -\sum_{C=1}^{2}R(E_C,\nabla\!\!\!/_{E_C}E_A,\underline{L},E_B) - \sum_{C=1}^{2}R(E_C,E_A,\underline{L},\nabla\!\!\!/_{E_C}E_B)\\
&= \sum_{C=1}^{2}\epsilon_{CA}\nabla\!\!\!/_C{}^*\underline{\beta}_B = \delta_{AB}\sum_{C=1}^{2}\nabla\!\!\!/_C\underline{\beta}_C - \nabla\!\!\!/_B\underline{\beta}_A
\end{aligned}$$

$$\begin{aligned}
&\underline{L}\cdot(\rho\,\delta_{AB}+\sigma\,\epsilon_{AB}) + R(L,\Pi\nabla_{\underline{L}}E_A,\underline{L},E_B) + R(L,E_A,\underline{L},\Pi\nabla_{\underline{L}}E_B)\\
&= \delta_{AB}\,\underline{D}\rho + \epsilon_{AB}\,\underline{D}\sigma
\end{aligned}$$

$$\begin{aligned}
L\cdot\underline{\alpha}_{AB} &= (D\underline{\alpha})_{AB} + [L,E_A]^C\underline{\alpha}_{CB} + [L,E_B]^C\underline{\alpha}_{AC}\\
&= (D\underline{\alpha})_{AB} - \sum_{C=1}^{2}\chi_{AC}\,\underline{\alpha}_{CB} - \sum_{C=1}^{2}\chi_{BC}\,\underline{\alpha}_{CA}\\
&= (D\underline{\alpha})_{AB} - \mathrm{tr}\,\chi\,\underline{\alpha}_{AB} - \delta_{AB}\,\hat{\chi}\cdot\underline{\alpha}
\end{aligned}$$

Here,

$$\sum_{C=1}^{2}\hat{\chi}_{AC}\,\underline{\alpha}_{CB} + \sum_{C=1}^{2}\hat{\chi}_{BC}\,\underline{\alpha}_{CA} = \delta_{AB}\,\hat{\chi}\cdot\underline{\alpha}$$

$$\hat{\chi}\cdot\underline{\alpha} = \sum_{A,C=1}^{2}\hat{\chi}_{AC}\,\underline{\alpha}_{CA}$$

since $\hat{\chi}$ and $\underline{\alpha}$ are trace-free and symmetric, and for the following we introduce the left and right Hodge duals,

$$^*\hat{\underline{\chi}}_{AB} = \sum_{C=1}^{2}\epsilon_{AC}\,\hat{\underline{\chi}}_{CB} \qquad \hat{\underline{\chi}}^*_{AB} = \sum_{C=1}^{2}\hat{\underline{\chi}}_{AC}\,\epsilon_{CB}$$

which are also trace-free and symmetric so that $^*\hat{\underline{\chi}} = -\hat{\underline{\chi}}^*$, and the trace-free part of the symmetrized covariant derivative

$$(\nabla\!\!\!/\hat{\otimes}\underline{\beta})_{AB} = \nabla\!\!\!/_A\underline{\beta}_B + \nabla\!\!\!/_B\underline{\beta}_A - \delta_{AB}\,\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\beta}\,,$$

and similarly

$$(\zeta\hat{\otimes}\underline{\beta})_{AB} = \zeta_A\,\underline{\beta}_B + \zeta_B\,\underline{\beta}_A - \delta_{AB}\,\zeta\cdot\underline{\beta}\,,$$

$$\zeta\cdot\underline{\beta} = \sum_{C=1}^{2}\zeta_C\,\underline{\beta}_C\,.$$

Then I sum up all terms (88)-(99) and conclude from (85):

$$\begin{aligned}
&\frac{1}{2}\,(D\underline{\alpha})_{AB} - \frac{1}{4}\,\mathrm{tr}\,\chi\,\underline{\alpha}_{AB} - \frac{1}{2}\,\delta_{AB}\,\hat{\chi}\cdot\underline{\alpha}\\
&\qquad - \frac{1}{2}\,\delta_{AB}\,(\underline{D}\rho + \frac{3}{2}\,\mathrm{tr}\,\underline{\chi}\,\rho) - \frac{1}{2}\,\epsilon_{AB}\,(\underline{D}\sigma + \frac{3}{2}\,\mathrm{tr}\,\underline{\chi}\,\sigma)\\
&= \; -(\nabla\!\!\!/\hat{\otimes}\underline{\beta})_{AB} + \nabla\!\!\!/_A\underline{\beta}_B\\
&\qquad - \frac{3}{2}\,(\hat{\underline{\chi}}_{AB}\,\rho - {}^*\hat{\underline{\chi}}_{AB}\,\sigma) + \frac{1}{2}\sum_{C=1}^{2}\hat{\chi}_{AC}\,\underline{\alpha}_{CB}\\
&\qquad + 2\,(\zeta\hat{\otimes}\underline{\beta})_{AB} + \zeta_A\,\underline{\beta}_B + (\underline{\xi}\hat{\otimes}\beta)_{AB} - 2\,\underline{\xi}_B\,\beta_A
\end{aligned}$$

By symmetrizing in A and B I obtain:

$$\begin{aligned}
&(D\underline{\alpha})_{AB} - \frac{1}{2}\,\mathrm{tr}\,\chi\,\underline{\alpha}_{AB} - \delta_{AB}\,\hat{\chi}\cdot\underline{\alpha} - \delta_{AB}\,(\underline{D}\rho + \frac{3}{2}\,\mathrm{tr}\,\underline{\chi}\,\rho)\\
&= \; -(\nabla\!\!\!/\hat{\otimes}\underline{\beta})_{AB} + \delta_{AB}\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta}\\
&\qquad - 3\,(\hat{\underline{\chi}}_{AB}\,\rho - {}^*\hat{\underline{\chi}}_{AB}\,\sigma) + \frac{1}{2}\,\delta_{AB}\,\hat{\chi}\cdot\underline{\alpha}\\
&\qquad + 5\,(\zeta\hat{\otimes}\underline{\beta})_{AB} + \delta_{AB}\,\zeta\cdot\underline{\beta} - 2\,\delta_{AB}\,\underline{\xi}\cdot\beta
\end{aligned}$$

Without proof, I now infer, from another component of (83):

$$\underline{D}\rho + \frac{3}{2}\,\mathrm{tr}\,\underline{\chi}\,\rho = -\mathrm{d}\!\!/\mathrm{iv}\,\underline{\beta} - \frac{1}{2}\,\hat{\chi}\cdot\underline{\alpha} - \zeta\cdot\underline{\beta} + 2\,\underline{\xi}\cdot\beta$$

We are then left with:

$$\begin{aligned}
&D\underline{\alpha} - \frac{1}{2}\,\mathrm{tr}\,\chi\,\underline{\alpha} =\\
&\quad = \; -\nabla\!\!\!/\hat{\otimes}\underline{\beta} - 3\,(\hat{\underline{\chi}}\,\rho - {}^*\hat{\underline{\chi}}\,\sigma) + 5\,\zeta\hat{\otimes}\underline{\beta} + \delta\,\hat{\chi}\cdot\underline{\alpha}
\end{aligned}$$

This ensures the integrability of (49):

$$\begin{aligned}
&|-\overline{\mathrm{tr}\,\chi}\,r\underline{\alpha} + D(r\underline{\alpha})\,|_\gamma \le\\
&\le \; \left|\frac{r}{2}\,(\mathrm{tr}\,\chi - \overline{\mathrm{tr}\,\chi})\,\underline{\alpha}\right|_\gamma + \left|r\,\nabla\!\!\!/\hat{\otimes}\underline{\beta}\right|_\gamma\\
&\qquad +3\left|r\,\hat{\underline{\chi}}\,\rho\right|_\gamma + 3\left|r\,{}^*\hat{\underline{\chi}}\,\sigma\right|_\gamma + 5\left|r\,\zeta\hat{\otimes}\underline{\beta}\right|_\gamma + 2|r\,\hat{\chi}\cdot\underline{\alpha}|\\
&= \; \mathcal{O}(r^{-2})
\end{aligned}$$

Q.E.D.

We now proceed to the **limiting equations** on C_*^- (these are equations on S^2):

In the limit (45), using (57), (59), (55), (52) and (75), the null Codazzi equation (30) becomes

$$\overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}}\ \Sigma = \frac{1}{2}\overset{\circ}{\nabla\!\!\!/} H + Z\,, \tag{100}$$

(multiply (30) by r^2, pull back by Φ_u and take the limit $r_0^* \to \infty$; here $\circ$ marks derivatives w.r.t. $\overset{\circ}{\gamma}$). Similarly, using (58), (59), (53) and (80), the conjugate null Codazzi equation (42) becomes

$$\overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}}\ \Xi = B\,, \tag{101}$$

(multiply (42) by r, and proceed as above). In the limit (45) the Hodge system ((31),(44)) for ζ becomes,

$$\begin{aligned} \overset{\circ}{\mathrm{c}\!\!/\mathrm{url}}\, Z &= Q - \frac{1}{2}\Sigma \wedge \Xi & (102)\\ \overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}}\ Z &= \underline{N} + P - \frac{1}{2}\Sigma \cdot \Xi\,, & (103) \end{aligned}$$

(multiply both (31) and (44) by r^3, pull back by Φ_u and take the limit $r_0^* \to \infty$; note that $\mathrm{c}\!\!/\mathrm{url}\,\zeta$ and $\mathrm{d}\!\!/\mathrm{iv}\,\zeta$ are functions of order r^{-3} since ζ is a 1-form of order r^{-2}), where (59), (57), (58) and (82), (81), and (70) have been used.

These are the limits of the elliptic equations. We are now concerned with the limits of the propagation equations along C_*^-. These, namely the propagation equations along the integral curves of $\underline{L}$, are not derived in part I – however, while $\hat{\chi}$ satisfies

$$\underline{\hat{D}}\hat{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,\hat{\chi} = -\frac{1}{2}\mathrm{tr}\,\chi\,\underline{\hat{\chi}} + \nabla\!\!\!/\hat{\otimes}\zeta + \zeta\hat{\otimes}\zeta\,, \tag{104}$$

for $\underline{\hat{\chi}}$, one finds analogously to (23) that

$$\underline{\hat{D}}\underline{\hat{\chi}} = -\underline{\alpha}\,. \tag{105}$$

Since in view of (51) and the fact that Σ is trace-free relative to $\overset{\circ}{\gamma}$,

$$\begin{aligned} \lim_{r_0^*\to\infty} {\Phi_u}^*(\underline{\hat{D}}\hat{\chi}) &\overset{(47)}{=} \lim_{r_0^*\to\infty} (\underline{L}\cdot u)\frac{\hat{\partial}}{\partial u}{\Phi_u}^*\hat{\chi}\\ &= \lim_{r_0^*\to\infty} (-2\underline{L}\cdot r_u^*)\frac{\partial}{\partial u}{\Phi_u}^*\hat{\chi}\\ &\overset{(53)}{=} 2\frac{\partial \Sigma}{\partial u}\,, \end{aligned}$$

and similarly for $\underline{\hat{\chi}}$, these equations simply become in the limit (45)

$$2\,\frac{\partial\Sigma}{\partial u} = -\Xi\,, \tag{106}$$

and

$$2\,\frac{\partial\Xi}{\partial u} = -A\,, \tag{107}$$

(multiply (105) by r^{-1}, pull back (104) and (105) by Φ_u and take the limit $r_0^* \to \infty$), where also (79) is used.

Finally, recalling that the Bondi mass (72) is only a function of u, we are interested in an asymptotic law of the following form:

Proposition 8. *$\underline{N}$ is a monotonously decreasing function of u,*

$$\frac{\partial \underline{N}}{\partial u} = -\frac{1}{4}\,|\Xi|^2\,. \tag{108}$$

This is in fact the limit of the rescaled propagation equation for $\underline{\mu}$ along C_*^-.

As just indicated we now obtain

$$\frac{\partial M}{\partial u} = \frac{1}{8\pi}\int_{S^2}\frac{\partial \underline{N}}{\partial u}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}} = -\frac{1}{32\pi}\int_{S^2}|\Xi|^2\,\mathrm{d}\mu_{\overset{\circ}{\gamma}} \tag{109}$$

the *Bondi mass loss formula.* Thus we interpret

$$\frac{1}{32\pi}\,|\Xi|^2(\xi) \tag{110}$$

as the *gravitational power radiated to infinity* (at a given retarded time u) in a given direction $\xi \in S^2$, per unit solid angle.

Remark 6. For the identification of u with retarded time one must remember that the limit $r_0^* \to \infty$ is taken *first*, hence deviations of the incoming null geodesics from the timelike geodesics initiating on S_0^* can be neglected, (see figure 7). In particular the limits $|u| \to \infty$, which are taken *subsequent* to $r_0^* \to \infty$, *do not* correspond to the whole generators of C_*^- but rather small segments thereof around S_0^*, (for which u remains small in comparison to r_0^*). Furthermore the surfaces C_u^+ as labeled by (39) eventually become equally spaced[10] (as $r_0^* \to \infty$), taken for granted in the above.

The integrated power (110) over retarded time,

$$\frac{F}{4\pi}(\xi) \tag{111}$$

is then the *total gravitational energy radiated to infinity* in a given direction $\xi \in S^2$, per unit solid angle, where F is the following function on S^2:

$$F(\xi) = \frac{1}{8}\int_{-\infty}^{\infty}|\Xi|^2(\xi)\,\mathrm{d}u\,, \quad (\xi \in S^2). \tag{112}$$

[10]The role of u is conceptually better understood as *optical function* as introduced more generally in [ChKl-Stability].

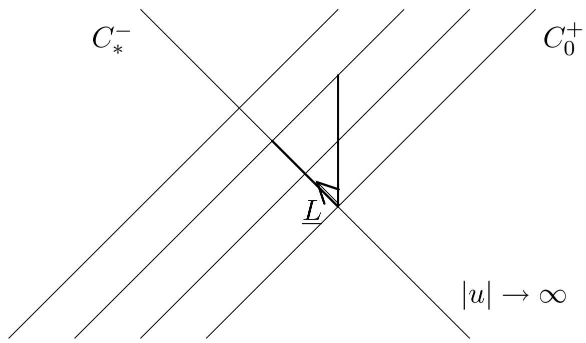

Figure 7: Identification of u with retarded time after the limit $r_0^* \to \infty$ is taken.

The existence of (112) relies, like the other **limits as $|u| \to \infty$** of the quantities discussed below, on the *Main Theorem* in [ChKl-Stability].

Proposition 9. *As $|u|$ tends to infinity,*

$$|\,\underline{\alpha}\,|_\gamma = \mathcal{O}(|u|^{-\frac{5}{2}}) \tag{113}$$

$$\left|\,\underline{\beta}\,\right|_\gamma = \mathcal{O}(|u|^{-\frac{3}{2}}) \tag{114}$$

$$|\,\sigma\,|_\gamma = \mathcal{O}(|u|^{-\frac{1}{2}})\,. \tag{115}$$

From equation (101),

$$\overset{\circ}{\not{d}\text{iv}}\ \Xi = B$$

and the fact (114) we see that also

$$\Xi = \mathcal{O}(|u|^{-\frac{3}{2}}) \qquad (|u| \to \infty)\,, \tag{116}$$

so Ξ is integrable and (112) indeed exists; moreover it results from (108) that

$$\underline{N}^+ - \underline{N}^- = -2F\,, \tag{117}$$

(here $\underline{N}^+$, $\underline{N}^-$ denote the limits of $\underline{N}$ as $u \to \infty$, $u \to -\infty$ respectively).

The existence of the *asymptotic shear* is an immediate consequence of (106) together with (116):

$$\Sigma^+ - \Sigma^- = -\frac{1}{2}\int_{-\infty}^{\infty} \Xi\,\mathrm{d}u\,.$$

To determine its explicit form we first consider the Z-system ((102),(103)). Integrating these equations over S^2 we obtain

$$\overline{Q} = \frac{1}{2}\overline{\Sigma \wedge \Xi}$$

$$\overline{P} = -\underline{\overline{N}} + \frac{1}{2}\overline{\Sigma \cdot \Xi}\,,$$

from which the limits of the mean asymptotic curvature are seen to be

$$\overline{Q}^{\pm} = 0 \tag{118}$$

$$\overline{P}^{+} - \overline{P}^{-} = -\underline{\overline{N}}^{+} + \underline{\overline{N}}^{-} = -2(M^{+} - M^{-}) = 2|M^{+} - M^{-}|\,, \tag{119}$$

since,

$$0 \leq M^{+} \leq M^{-}\,, \tag{120}$$

(the first inequality is generally true[11], and the second expresses (109)). While

$$(Q - \overline{Q})^{\pm} \overset{(118)}{=} Q^{\pm} \overset{(115)}{=} 0 \tag{121}$$

is apparently always true, this is *not* in general the case for P; however here I make the **simplifying assumption** that also

$$(P - \overline{P})^{\pm} = 0\,. \tag{122}$$

Remark 7. Consider a number of (gravitating and moving) masses in the Newtonian theory (for example the bodies constituting the source); these have a center of mass frame, and relative velocities w.r.t. this frame. Set in the context of general relativity (122) holds if and only if the final center of mass frame is at rest relative to the initial center of mass frame and moreover the initial and final relative velocities w.r.t. the corresponding frames are negligible. While in the absence of exterior forces the final center of mass frame will always be at rest relative to the initial center of mass frame in the Newtonian theory, a considerable non-zero relative velocity may arise in General Relativity from the recoil due to the radiation of linear momentum to infinity. The physically most interesting case is that where the initial velocities are negligible, then the initial data on a maximal spacelike hypersurface of vanishing linear momentum is strongly asymptotically flat, and $(P - \overline{P})^{-} = 0$. In general $(P - \overline{P})^{-}$ and $(P - \overline{P})^{+}$ are determined from boosted Schwarzschild solutions, (see [Ch-Exp]).

Then

$$\begin{aligned}
\overset{\circ}{\not{\mathrm{curl}}}\,(Z^{+} - Z^{-}) &\overset{(121)}{=} 0 \\
\overset{\circ}{\not{\mathrm{div}}}\,(Z^{+} - Z^{-}) &\overset{(122)}{=} \underline{N}^{+} + \overline{P}^{+} - \underline{N}^{-} - \overline{P}^{-} \\
&\overset{(117),(119)}{=} -2F + 2|M^{+} - M^{-}| = -2(F - \overline{F})\,,
\end{aligned}$$

and we conclude that there is a function ϕ on S^2 such that

$$Z^{+} - Z^{-} = \overset{\circ}{\not\nabla}\phi \tag{123}$$

$$\overset{\circ}{\not\Delta}\phi = -2(F - \overline{F})\,. \tag{124}$$

[11] Here it is referred to the *Positive Energy Theorem*, see [Ch-I].

Thus we learn that

$$\overset{\circ}{\text{d\!/iv}}\;(\Sigma^+ - \Sigma^-) \overset{(100)}{=} \overset{\circ}{\nabla\!\!\!/}\,\phi\,, \tag{125}$$

where it is used that

$$H^+ - H^- \overset{(56)}{=} 0\,.$$

Consider the divergence operator in (125) acting on trace-free symmetric 2-covariant tensorfields on the sphere. The adjoint operator is found as follows:

$$\begin{aligned}\int_{S^2}(\xi, \overset{\circ}{\text{d\!/iv}}\;\theta)\,\mathrm{d}\mu_{\overset{\circ}{\gamma}} &= \int_{S^2}\xi^a\;\overset{\circ}{\nabla\!\!\!/}{}^b\theta_{ab}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}}\\ &= -\int_{S^2}\overset{\circ}{\nabla\!\!\!/}{}^b\xi^a\theta_{ab}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}}\\ &= -\frac{1}{2}\int_{S^2}\{\overset{\circ}{\nabla\!\!\!/}{}^a\xi^b + \overset{\circ}{\nabla\!\!\!/}{}^b\xi^a - (\overset{\circ}{\gamma}{}^{-1})^{ab}\overset{\circ}{\text{d\!/iv}}\;\xi\}\theta_{ab}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}}\end{aligned}$$

The curly bracket is simply the trace-free part of the symmetrized covariant derivative:

$$(\overset{\circ}{\nabla\!\!\!/}\,\hat{\otimes}\xi)_{ab} = \overset{\circ}{\nabla\!\!\!/}{}_a\xi_b + \overset{\circ}{\nabla\!\!\!/}{}_b\xi_a - \overset{\circ}{\gamma}_{ab}\;\overset{\circ}{\text{d\!/iv}}\;\xi\,.$$

We conclude that

$$\overset{\circ}{\text{d\!/iv}}{}^*\,\xi = -\frac{1}{2}\,\overset{\circ}{\nabla\!\!\!/}\,\hat{\otimes}\xi = -\frac{1}{2}\,\widehat{\pounds_{\xi^\#}\,\overset{\circ}{\gamma}}\,,$$

where $\xi^\#$ denotes the vectorfield corresponding to the 1-form ξ, because, (by Leibniz rule, metric compatibility and symmetry of connection)

$$(\pounds_{\xi^\#}\,\overset{\circ}{\gamma})(X,Y) = \overset{\circ}{\gamma}\,(\nabla_X\xi^\#, Y) + \overset{\circ}{\gamma}\,(X, \nabla_Y\xi^\#)\,.$$

Since the gradient of ϕ in (125) is apparently in the image of the divergence operator, it must be anulled by the kernel of its adjoint, (by general principle), which are the conformal Killing fields of the sphere:

$$\ker\overset{\circ}{\text{d\!/iv}}{}^* = \left\{X \in \mathcal{X}(S^2) : \widehat{\pounds_X\,\overset{\circ}{\gamma}} = 0\right\}\,.$$

So we must have

$$0 = \int_{S^2} X\cdot\overset{\circ}{\nabla\!\!\!/}\,\phi\;\mathrm{d}\mu_{\overset{\circ}{\gamma}} = -\int_{S^2}\overset{\circ}{\text{d\!/iv}}\;X\,\phi\;\mathrm{d}\mu_{\overset{\circ}{\gamma}} \qquad \forall X \in \ker\overset{\circ}{\text{d\!/iv}}{}^*\,,$$

hence

$$\phi_{(1)} = 0\,, \tag{126}$$

where subscript (1) denotes the projection on the first eigenspace of the spherical Laplacian $\overset{\circ}{\triangle\!\!\!/}$, because the divergences of the conformal Killing fields of the sphere are precisely the eigenfunctions to the first eigenvalue of $\overset{\circ}{\Delta}$ (eigenvalue -2, multiplicity 3).

The integrability condition (126) is seen from (124) to be equivalent to

$$F_{(1)} = 0\,, \tag{127}$$

once one notes that $F_{(0)} = \overline{F}$. Having in mind the interpretation of F as total energy radiated to infinity (recall (111)), the explicit formula for the projection

$$F_{(1)}(\xi) = \frac{3}{4\pi}\int_{S^2}\langle\xi,\xi'\rangle F(\xi')\,\mathrm{d}\mu_{\overset{\circ}{\gamma}}(\xi')$$

shows that (127) states quite remarkably the vanishing of the total linear momentum radiated to infinity in any given direction.

Proposition. *Under the condition (127) the system ((125),(124)) has the solution*

$$(\Sigma^+ - \Sigma^-)_\xi(X,Y) =$$

$$= -\frac{1}{2\pi}\int_{S^2}(F-\overline{F})(\xi')\frac{\langle X,\xi'\rangle\langle Y,\xi'\rangle - \frac{1}{2}\langle X,Y\rangle|\Pi\xi'|^2}{1-\langle\xi,\xi'\rangle}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}}(\xi')\,, \tag{128}$$

$$(X,Y\in T_\xi S^2,\ \xi\in S^2)\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product in $\boldsymbol{R}^3$, and Π is the projection on the plane orthogonal to ξ through the origin, (see figure 8).

Note that to $(\Sigma^+-\Sigma^-)_\xi$, the asymptotic shear in the direction of observation $\xi\in S^2$, does not only contribute $F(\xi)$, the total energy radiated to infinity in the direction ξ, but there is a contribution from *each* $F(\xi')$, $\xi'\in S^2$.

Notation. On S^2 we introduce local coordinates (ϑ,φ) (polar coordinates) so that each $\xi\in S^2\subset\boldsymbol{R}^3$ identified with the corresponding element in $T_0\boldsymbol{R}^3$ is written as

$$\xi = \sin\vartheta\cos\varphi\frac{\partial}{\partial x} + \sin\vartheta\sin\varphi\frac{\partial}{\partial y} + \cos\vartheta\frac{\partial}{\partial z}\,, \tag{129}$$

then the metric takes the form

$$\overset{\circ}{\gamma} = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\,.$$

Moreover, given $X\in T_\xi S^2$,

$$X = X^\vartheta\frac{\partial}{\partial\vartheta}|_\xi + X^\varphi\frac{\partial}{\partial\varphi}|_\xi\,. \tag{130}$$

What we mean by the inner products in (128) is that ξ and ξ' are taken in the form (129), while X and Y as in (130) are first expanded in terms of

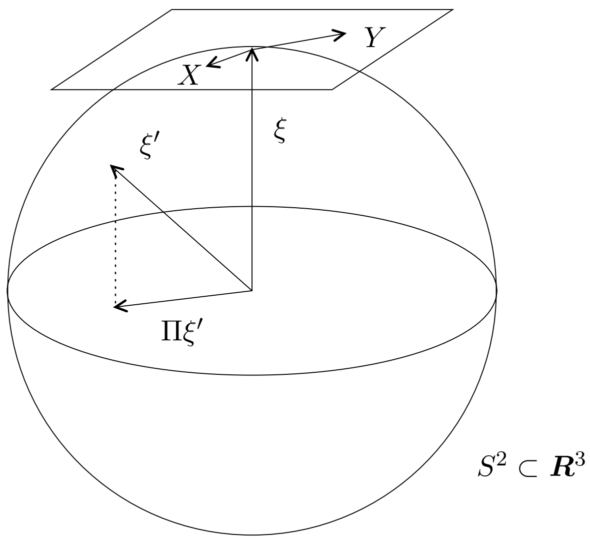

Figure 8: Quantities in the integration of the asymptotic shear.

$(\frac{\partial}{\partial x}|_\xi, \frac{\partial}{\partial y}|_\xi, \frac{\partial}{\partial z}|_\xi)$ and then thought of as vectors attached to the origin. We have

$$\begin{aligned}
\frac{\partial}{\partial \vartheta}|_\xi &= \cos\vartheta\cos\varphi\frac{\partial}{\partial x}|_\xi + \cos\vartheta\sin\varphi\frac{\partial}{\partial y}|_\xi - \sin\vartheta\frac{\partial}{\partial z}|_\xi \\
\frac{\partial}{\partial \varphi}|_\xi &= -\sin\vartheta\sin\varphi\frac{\partial}{\partial x}|_\xi + \sin\vartheta\cos\varphi\frac{\partial}{\partial y}|_\xi\,,
\end{aligned}$$

so for example

$$\langle\frac{\partial}{\partial\vartheta}|_\xi, \xi'\rangle = \cos\vartheta\cos\varphi\sin\vartheta'\cos\varphi' + \cos\vartheta\sin\varphi\sin\vartheta'\sin\varphi' - \sin\vartheta\cos\vartheta'\,.$$

Note that

$$|\Pi\xi'|^2 = 1 - \langle\xi', \xi\rangle^2\,,$$

and

$$\sqrt{\det \overset{\circ}{\gamma}} = \sin\vartheta\,.$$

Remark 8. If the simplifying assumption (122) is not made (124) reads

$$\overset{\circ}{\triangle}\phi = (P - \overline{P})^+ - (P - \overline{P})^- - 2(F - \overline{F})$$

which leads to an additional "linear" term[12] in the solution (128), (see [Ch-Exp]). Moreover the integrability condition is not (127) but

$$P^+_{(1)} - P^-_{(1)} - 2F_{(1)} = 0\,,$$

which expresses the conservation of linear momentum.

[12]This term is also known in the linearized theory.

III. Gravitational Wave Experiment and the Memory Effect

A gravitational wave experiment consists of three test masses, a reference mass m_0 and two masses m_1, m_2, at equal distances d_0 and right angles from m_0, (see figure 9). The masses are initially at rest, but their motion is free in the common horizontal plane. If the experiment is performed in space the motion of the masses is free anyway, and if the experiment is performed on earth, this is achieved by suspending the test masses by pendulums with sufficiently large periods.

The passage of a gravitational wave train (a sequence of wave fronts as elaborated in part I and II) causes, as is shown below, a *permanent displacement* of the test masses in the horizontal plane, (under the assumptions stated below).

However, the practical measurement can only determine differences in light travel time between m_0 and m_1, and m_0 and m_2, by splitting an incident laser beam at m_0, and observing the interference pattern of the reflected beams from m_1 and m_2, at m_0; any difference in the light travel times results in a difference of phase, (see figure 10). To understand the measurement of light travel time and its relation to distance, we think of the motion of the test masses as follows:

Let Γ_0 be the timelike geodesic in spacetime described by the reference mass m_0, parametrized by arc length t (proper time). We choose $t = 0$ at a point which corresponds to the initial setup of the experiment, where the test masses m_1, m_2 are at rest relative to m_0. Let T be the unit future-directed tangent vector field to Γ_0, $T(\Gamma_0(t)) = \dot{\Gamma}_0(t)$, and let for each t

$$\Sigma_t = \Big\{ X \in T_{\Gamma_0(t)}\mathcal{M} : \; g(X,T) = 0 \Big\} \;,$$

a hyperplane in $T_{\Gamma_0(t)}\mathcal{M}$. It can be chosen unit vectors $E_1, E_2 \in \Sigma_0$ pointing in the direction of m_1, m_2, respectively,

$$\exp_{\Gamma_0(0)}(d_0 E_A) = \Gamma_A(0) \,, \qquad (A = 1,2) \,,$$

where Γ_1, Γ_2 are the nearby timelike geodesics described by m_1, m_2, respectively, (see figure 11). As for any nearby geodesic Γ parametrized by arc length we choose the origin of the proper time of Γ_1, Γ_2 on the spacelike hypersurface

$$H_0 = \exp_{\Gamma_0(0)} \Sigma_0 \,.$$

Next we complement E_1, E_2 by a vector $E_3 \in \Sigma_0$ to an orthonormal basis for Σ_0, and propagate it along Γ_0 by parallel transport:

$$\nabla_T E_i = 0 \,, \quad (i = 1,2,3) \,. \tag{131}$$

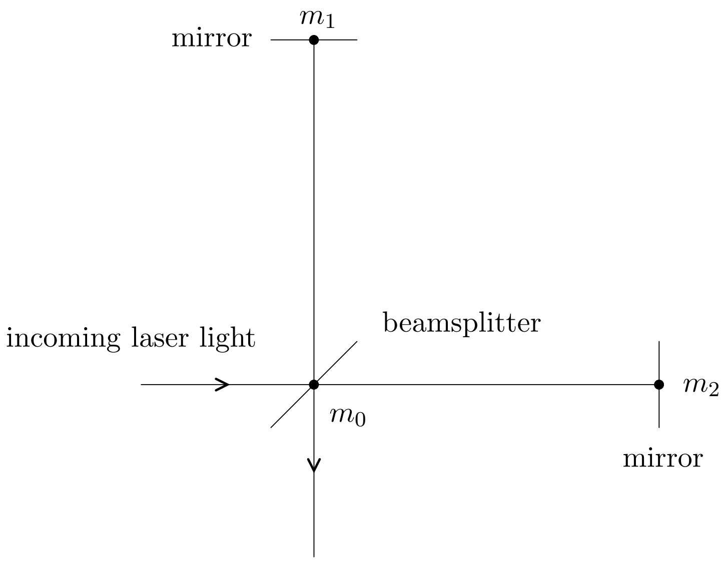

Figure 9: Arrangement of test masses in the horizontal plane, and schematics of a laser interferometer gravitational wave detector.

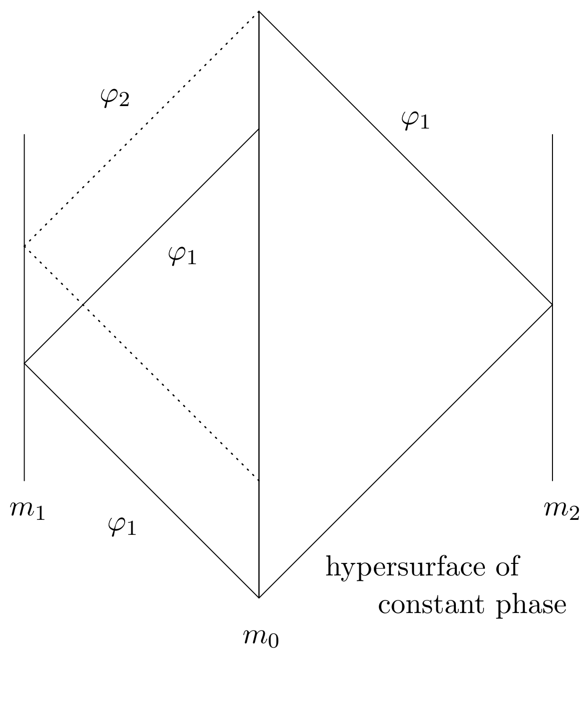

Figure 10: Interferometer measurement in (1+1)-spacetime. Shown are (null) hypersurfaces of two different phases, (φ_1 and φ_2).

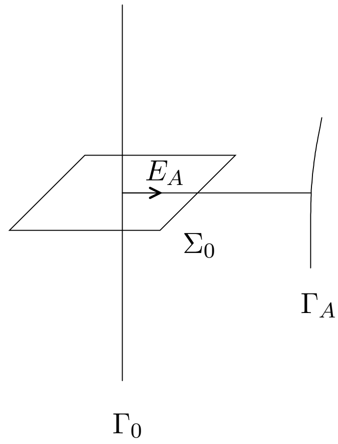

Figure 11: The reference geodesic Γ_0 in spacetime.

Since

$$T \cdot g(T, E_i) = g(\nabla_T T, E_i) + g(T, \nabla_T E_i) = 0\,,$$

this is an orthonormal basis $(E_i(t) : \; i = 1, 2, 3)$ for Σ_t. Now any event p in the neighborhood of Γ_0 can be assigned the coordinates (t, x^1, x^2, x^3) if

$$p = \exp_{\Gamma_0(t)} X\,, \tag{132}$$

where $X = \sum_{i=1}^{3} x^i E_i \in \Sigma_t$. Then $p \in H_t = \exp_{\Gamma_0(t)} \Sigma_t$, and the distance d (arc length) of p from Γ_0 is

$$d = |X| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,.$$

In the following passage I derive the **equation of motion** of the nearby masses in these coordinates. This is only sensible under a certain assumption on the dimension of the detector in comparison to the period of the expected gravitational wave, to be discussed afterwards.

First we extend T to H_0 by parallel transport: For each $p = \exp_{\Gamma_0(0)} X \in H_0$, we parallel transport $T(\Gamma_0(0))$ along the geodesic initiating at $\Gamma_0(0)$ with initial tangent vector $X \in \Sigma_0$ to p. Secondly T is extended along the timelike geodesics initiating on H_0 with initial tangent vectors $T|_{H_0}$ to be the tangent vector field along these geodesics. Lastly let ϕ_t be the 1-parameter group generated by T.

Consider for a given $X \in \Sigma_0$ the timelike geodesics initiating on the above spacelike geodesic, that is on

$${}^{(X)}K_0(\lambda) = \exp_{\Gamma_0(0)}(\lambda X)\,, \qquad (\lambda \in [0, 1])\,.$$

If $X(t) \in \Sigma_t$ is to describe the geodesic initiating at ${}^{(X)}K_0(1)$ it must be tangent to the curve

$$K_t = \phi_t({}^{(X)}K_0)\,,$$

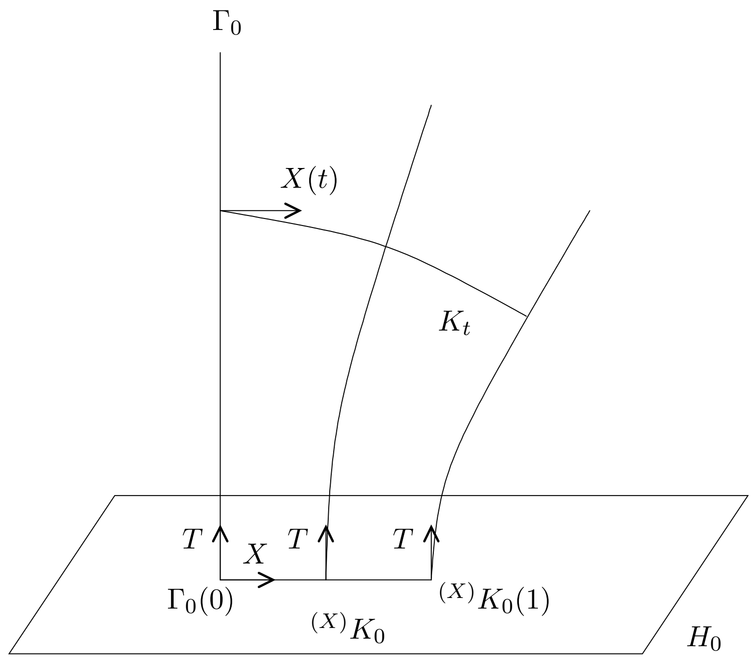

Figure 12: Construction of the displacement vector.

(see figure 12), therefore

$$X(t) = \mathrm{d}\phi_t \cdot X \,.$$

Since

$$(\phi_{-t\,*}X(t))(\Gamma_0(0)) = \mathrm{d}\phi_{-t}(\Gamma_0(t)) \cdot X(t) = X \,,$$

it follows that

$$[T, X] = \pounds_T X = \lim_{t\to 0} t^{-1}(\phi_{-t*}X - X) = 0 \,. \tag{133}$$

On the one hand we have

$$\nabla_T X = \sum_{i=1}^{3}(T\cdot x^i)E_i + \sum_{i=1}^{3} x^i\,\nabla_T E_i \overset{(131)}{=} \sum_{i=1}^{3}\frac{\mathrm{d}x^i}{\mathrm{d}t}E_i \,,$$

and

$$\nabla_T^2 X = \nabla_T\nabla_T X = \sum_{i=1}^{3}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}E_i \,.$$

Note that

$$\nabla_X T|_{t=0} \overset{(133)}{=} \nabla_T X|_{t=0} = \sum_{i=1}^{3}\frac{\mathrm{d}x^i}{\mathrm{d}t}|_{t=0}\,E_i = 0 \,,$$

because the test masses are assumed to be initially at rest relative to m_0—this justifies that T was parallel transported along H_0.
On the other hand we have

$$\begin{aligned}\nabla_T^2 X &\overset{(133)}{=} \nabla_T \nabla_X T = \\ &\overset{(11)}{=} \nabla_X \nabla_T T + \nabla_{[T,X]} T + R(T,X) \cdot T \overset{(133)}{=} -R(X,T) \cdot T\,,\end{aligned}$$

($\nabla_T T = 0$ because Γ_0 is a geodesic). Since

$$g(R(X,T) \cdot T, T) = R(T,T,X,T) = 0\,,$$

the right hand side can be expanded in the basis ($E_i :\ i = 1, 2, 3$):

$$R(X,T) \cdot T = \sum_{i=1}^{3} \sum_{j=1}^{3} R(E_i, T, E_j, T)\, x^j\, E_i\,.$$

Setting $E_0 = T$ we conclude

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\sum_{j=1}^{3} R_{i0j0}\, x^j\,; \tag{134}$$

this is the *Jacobi equation*.

However, the curves K_t are *not* geodesics, in particular $K_t(1) \notin H_t$, so the curve

$$\exp_{\Gamma_0(t)} X(t)\,, \quad (t \geq 0)\,,$$

cannot describe the timelike geodesic initiating at ${}^{(X)}K_0(1)$ *exactly*, but the disagreement is negligible as long as the geodesic deviation remains small in comparison to the initial displacement.

Let us introduce the *period* τ of the gravitational wave (the time scale of significant variation of the curvature), then we read off from (134) that

$$\frac{\Delta d}{d_0} = \mathcal{O}(R\tau^2)\,, \tag{135}$$

where Δd is the resulting displacement of the test masses from their initial positions after the passage of the wave train. Thus in order for the geodesic deviation to be small on time intervals *of order* d_0 we must make the **assumption** that

$$\frac{d_0}{\tau} \ll 1\,, \tag{136}$$

(compare figure 13). Our interest in time intervals of that size stems from the fact that this is the order of the light travel time between the reference mass and the test masses: At a point p in the neighborhood of Γ_0 expressed as (132) the metric components in this coordinate system are

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(Rd^2)\,,$$

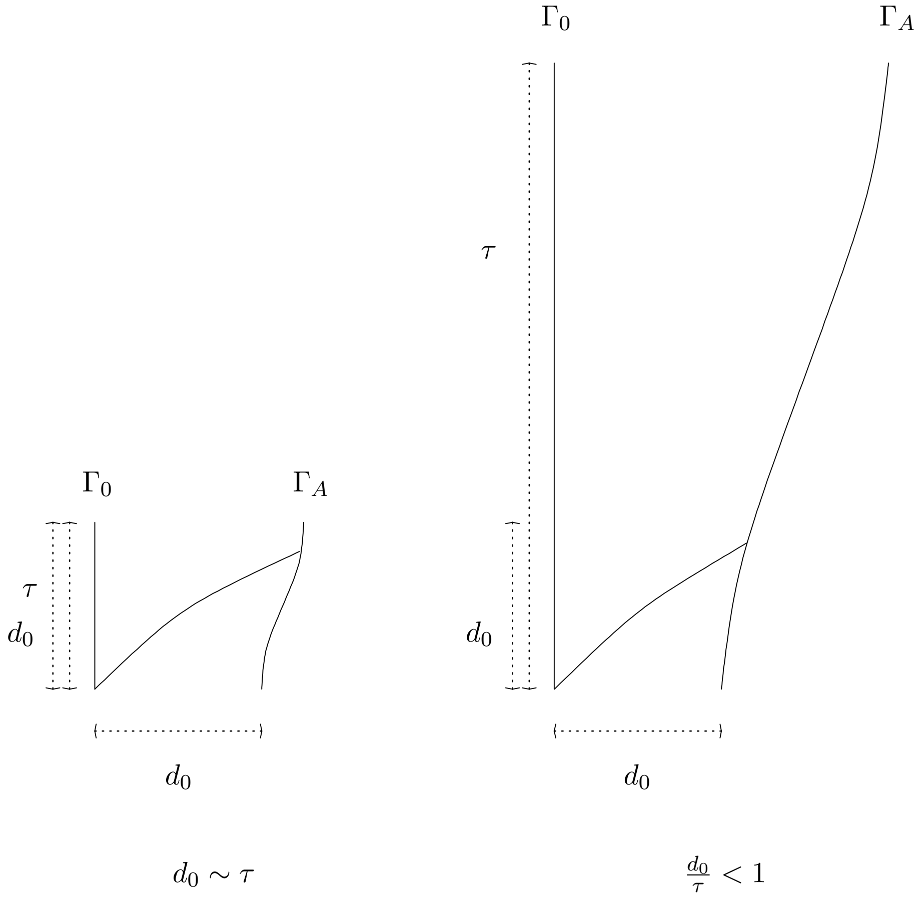

Figure 13: Geodesic deviation and its measurement by light travel time in the cases where d_0 is comparable to τ, and where d_0 is smaller than τ.

therefore the light travel times t_1, t_2 from m_0 to m_1, m_2, respectively satisfy

$$\frac{d_A}{t_A} = 1 + \mathcal{O}(Rd_0^2)\,.$$

So under the assumption (136) differences in light travel time accurately enough reflect differences in distance, (see also figure 13).

Although (136) suggests a favourably small extend of the experiment, it is quite the contrary from the experimental point of view, since also according to (134), or (135), the displacements will then become unmeasureably small.

Example. For a binary neutron star system $\tau \sim 10^{-3}\,\mathrm{sec} = 300\,\mathrm{km}$, while for the *VIRGO* detector $d_0 = 3\,\mathrm{km}$, so in this case

$$\frac{d_0}{\tau} = 10^{-2}\,.$$

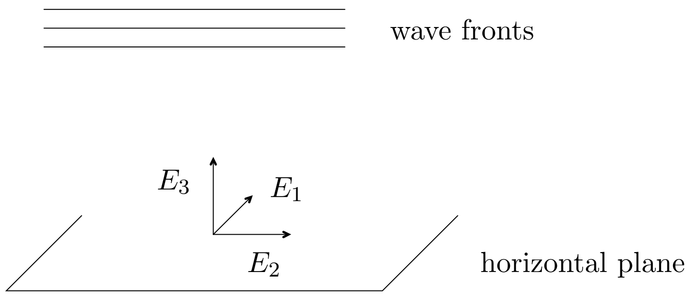

Figure 14: Orientation of the horizontal plane towards the source in space.

For simplicity let us now **suppose** that the waves are coming from the vertical, namely that the source is in the direction E_3. Here we choose an orientation for Σ_t such that (E_1, E_2, N) is a positive basis, where $N = -E_3$ is the normal to the wavefronts. Then (E_2, E_1, E_3) is also a positive basis, (see figure 14).

Recall the construction of the surface S_0^* in part I, (37) and figure 3, and also of the surfaces S_u^*, see figure 5. In this case the horizontal plane spanned by E_1, E_2 in Σ_0, more generally in Σ_t, is *tangent* to S_0^*, respectively to S_u^*, (in the limit $r_0^* \to \infty$ t and u are identified, compare Remark 6). Moreover the incoming and outgoing null normals on S_u^* coincide with

$$L = T + N = T - E_3 \qquad \underline{L} = T + E_3\,, \tag{137}$$

(see figure 15), or

$$T = \frac{1}{2}(\underline{L} + L) \qquad E_3 = \frac{1}{2}(\underline{L} - L)\,. \tag{138}$$

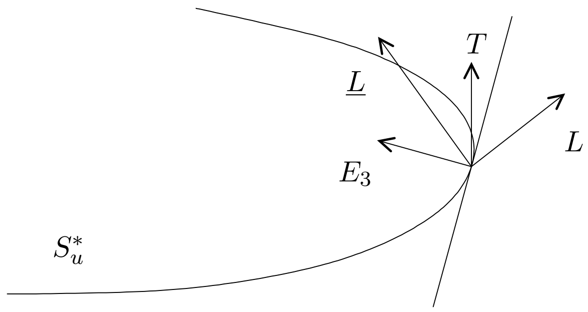

Figure 15: The relation of the vectors T, E_3 to the null normals, L, $\underline{L}$.

I now return to (134):
In the case $i = j = 3$,

$$R(E_3, T, E_3, T) = \frac{1}{4} R(\underline{L}, L, \underline{L}, L) = -\frac{1}{2} \sum_{A=1}^{2} R(\underline{L}, E_A, L, E_A)\,,$$

here recall (15) to write out the $(\underline{L}, L)$ component of (5). But

$$\begin{aligned} R(\underline{L}, E_A, L, E_B) + R(L, E_A, \underline{L}, E_B) &= \\ = 2 \sum_{C=1}^{2} R(E_C, E_A, E_C, E_B) = -2\rho \sum_{C=1}^{2} \epsilon_{CA}\epsilon_{CB} &= -2\,\rho\,\delta_{AB} \end{aligned} \tag{139}$$

and by the cyclic identity

$$\begin{aligned} R(\underline{L}, E_A, L, E_B) - R(L, E_A, \underline{L}, E_B) &= \\ = +R(\underline{L}, L, E_A, E_B) &= 2\,\sigma\,\epsilon_{AB}\,, \end{aligned} \tag{140}$$

so adding the two equations

$$R(\underline{L}, E_A, L, E_B) = -\rho\,\delta_{AB} + \sigma\,\epsilon_{AB}\,.$$

We conclude that

$$R(E_3, T, E_3, T) = \rho = \mathcal{O}(r^{-3})$$

according to Proposition 6.
The next case is $i = 3$, $j = A = 1, 2$ (or $i = A = 1, 2$, $j = 3$, which is the same because of the pair symmetry of the curvature tensor):

$$\begin{aligned} R(E_3, T, E_A, T) &= \frac{1}{4} R(\underline{L}, L, E_A, \underline{L}) + \frac{1}{4} R(\underline{L}, L, E_A, L) \\ &= \frac{1}{2}\,\underline{\beta}_A + \frac{1}{2}\,\beta_A = \mathcal{O}(r^{-2}) \end{aligned}$$

according to Proposition 6.
Finally there is the case $i = A = 1, 2$, $j = B = 1, 2$:

$$\begin{aligned} R(E_A, T, E_B, T) &= \frac{1}{4} R(E_A, \underline{L}, E_B, \underline{L}) + \frac{1}{4} R(E_A, \underline{L}, E_B, L) \\ &\quad + \frac{1}{4} R(E_A, L, E_B, \underline{L}) + \frac{1}{4} R(E_A, L, E_B, L) \\ &\overset{(139)}{=} \frac{1}{4}\,\underline{\alpha}_{AB} - \frac{1}{2}\,\rho\,\delta_{AB} + \frac{1}{4}\,\alpha_{AB} \\ &= \frac{1}{4}\,\underline{\alpha}_{AB} + \mathcal{O}(r^{-3})\,, \end{aligned}$$

where $\underline{\alpha}_{AB} = \mathcal{O}(r^{-1})$ is the leading order term according to Proposition 6.

This implies that the components of (134) are to leading order:

$$\begin{aligned} \frac{\mathrm{d}^2 x^3}{\mathrm{d}t^2} &= \mathcal{O}(r^{-2}) \\ \frac{\mathrm{d}^2 x^A}{\mathrm{d}t^2} &= -\frac{1}{4}\sum_{B=1}^{2} \underline{\alpha}_{AB}\, x^B + \mathcal{O}(r^{-2}) \end{aligned} \tag{141}$$

To make the one over r dependence explicit, we introduce a frame $(\tilde{E}_A : A = 1,2)$ on S^2:

$$r\, E_A = \mathrm{d}\Phi_u \cdot \tilde{E}_A\,, \quad (A = 1,2)\,,$$

where $\Phi_u : S^2 \to S^*_u$ is the diffeomorphism from part II. Then

$$\delta_{AB} = \gamma(E_A, E_B) = \Phi_u{}^*(\frac{1}{r^2}\gamma)(\tilde{E}_A, \tilde{E}_B)\,,$$

hence, by (50),

$$\overset{\circ}{\gamma}(\tilde{E}_A, \tilde{E}_B) = \delta_{AB}\,,$$

and $(\tilde{E}_A : A = 1,2)$ is an orthonormal frame for $(S^2, \overset{\circ}{\gamma})$. For the components of $\underline{\alpha}$,

$$\underline{\alpha}_{AB} = \frac{1}{r}(\Phi_u{}^*\frac{1}{r}\underline{\alpha})(\tilde{E}_A, \tilde{E}_B)\,,$$

recall the limit (79):

$$\lim_{r_0^* \to \infty} \Phi_u{}^*(r_u^{*\,-2+1}\underline{\alpha}) = A\,.$$

This means that

$$\underline{\alpha}_{AB} = \frac{1}{r} A_{AB} + \mathcal{O}(r^{-2})$$

where A_{AB} are the components of A in the frame $(\tilde{E}_A : A = 1,2)$ and (141) in fact reads for the test masses m_1, m_2,

$$\frac{\mathrm{d}^2 x^3_{(A)}}{\mathrm{d}t^2} = \mathcal{O}(r^{-2}) \tag{142}$$

$$\frac{\mathrm{d}^2 x^A_{(C)}}{\mathrm{d}t^2} = -\frac{1}{4r}\sum_{B=1}^{2} A_{AB}\, x^B_{(C)} + \mathcal{O}(r^{-2}) \tag{143}$$

where the i^{th} coordinate of m_A is denoted by $x^i_{(A)}$. The initial conditions are

$$x^i_{(A)}(0) = d_0\, \delta^i_A\,, \quad \dot{x}^i_{(A)}(0) = 0\,. \tag{144}$$

Remark. The initial conditions are set at $t = 0$ for the clarity of the construction, but exclude any tidal displacements due to wavefronts prior to that initiating on S_0 and propagating on C_0^+. Since the choice of S_0 is arbitrary there is no reason to do that and we impose (144) from now on not at $t = 0$, but as $t \to -\infty$.

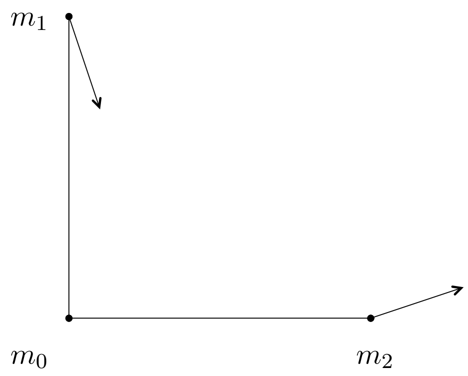

Figure 16: The instantaneous acceleration of m_2 is obtained from that of m_1 by counterclockwise rotation by 90° in the horizontal plane.

Since the fractional displacements are small, see (135), the initial values (144) can be substituted on the right hand side of (143) to arrive at the approximate equations of motion (which neglect all corrections of quadratic order in inverse distance from the source):

$$\frac{\mathrm{d}^2 x^3_{(A)}}{\mathrm{d}t^2} \cong 0 \tag{145}$$

$$\frac{\mathrm{d}^2 x^A_{(B)}}{\mathrm{d}t^2} \cong -\frac{d_0}{4r} A_{AB} \,. \tag{146}$$

Two **qualitative statements** can be made about the motion of the test masses in leading order: First *the motion is confined to the horizontal plane.* Secondly *the instantaneous acceleration of one test mass is obtained from that of the other by rotation by* 90°. While the first is clearly seen from (145), the second follows alone from the fact that (A_{AB}) is a symmetric trace-free 2×2 matrix and (146), (see figure 16):

$$\begin{aligned} \ddot{x}^1_{(2)} &= \ddot{x}^2_{(1)} \\ \ddot{x}^2_{(2)} &= -\ddot{x}^1_{(1)} \,. \end{aligned}$$

We now have everything at hand to integrate the equations (145) and (146); (recall also the identification of the retarded time u with the proper time t along Γ_0). The result of integrating (145) twice is simply

$$x^3_{(A)}(t) = 0 \,;$$

the masses remain on the horizontal plane. The velocities in the horizontal plane

$$\dot{x}^A_{(B)}(t) = \dot{x}^A_{(B)}(-\infty) - \int_{-\infty}^{t} \frac{d_0}{4r} A_{AB}(u) \, \mathrm{d}u$$

are in fact, in view of (107), and (116),

$$\dot{x}^A_{(B)}(t) = \frac{d_0}{2r}\,\Xi_{AB}(t)\,, \tag{147}$$

(the components of Ξ are given in the frame $(\tilde{E}_1, \tilde{E}_2)$). By (116) the test masses eventually come to rest,

$$\lim_{t\to\infty} \dot{x}^A_{(B)}(t) = 0\,.$$

At proper time t they are found at

$$x^A_{(B)}(t) = x^A_{(B)}(-\infty) + \int_{-\infty}^{t} \frac{d_0}{2r}\,\Xi_{AB}(u)\,\mathrm{d}u\,,$$

which is, using (106),

$$x^A_{(B)}(t) = d_0\,\delta^A{}_B - \frac{d_0}{r}(\Sigma_{AB}(t) - \Sigma^-_{AB})\,.$$

When the test masses come to rest, they have suffered a *permanent displacement* from their initial positions:

$$\triangle x^A_{(B)} = -\,\frac{d_0}{r}\,(\Sigma^+ - \Sigma^-)_{AB}\,, \tag{148}$$

for which an explicit formula is known, namely (128). This fact is naturally called the **memory effect**.

Remark 9. The permanent displacement (148) is a manifestation of the *nonlinear* nature of gravitation, i.e. of the nonlinearity of (5). For in the linearized theory, according to the *quadrupole formula*,

$$A \sim \frac{\mathrm{d}^4}{\mathrm{d}t^4}\,Q\,, \tag{149}$$

where Q is the quadrupole moment of the source in the Newtonian framework; for a stellar binary system this is simply

$$Q_{ij} = \sum_{\alpha=1}^{2} M_\alpha\, X^i_\alpha X^j_\alpha\,. \tag{150}$$

Using (107) and (106) as definitions this leads to

$$\triangle x \sim \frac{\mathrm{d}^2 Q}{\mathrm{d}t^2}(\infty) - \frac{\mathrm{d}^2 Q}{\mathrm{d}t^2}(-\infty)\,. \tag{151}$$

Since

$$\ddot{Q}_{ij} = 2\sum_{\alpha=1}^{2} M_\alpha\,\dot{X}^i_\alpha \dot{X}^j_\alpha + \sum_{\alpha=1}^{2} M_\alpha(\ddot{X}^i_\alpha X^j_\alpha + X^i_\alpha \ddot{X}^j_\alpha)$$

a permanent displacement is absent provided the astronomical bodies constituting the source are initially and finally at rest in the respectively initial and final center of mass frame, and the final center of mass frame is at rest relative to the initial center of mass frame. But this is precisely what the assumption (122) amounts to—and still the resulting displacement (148) is here nonzero. (More generally (151) vanishes if there is no overall change of the linear momenta of the constituent bodies. But (148) does *not* under precisely the same assumption, which is indeed the statement of the integrability condition (127).) Thus a nonlinear memory must be ascribed to every gravitational wave burst, independently of the circumstances under which it is produced. This is due to the cumulative contribution of the effective stress $|\Xi|^2$ of the gravitational waves themselves: the displacement (148) is determined by (128), an integration over the sphere with a certain kernel, where the function F is given by (112),

$$F = \frac{1}{8}\int_{-\infty}^{\infty} |\Xi|^2 \, \mathrm{d}u \,,$$

which is quadratic in Ξ.

Remark 10. The permanent displacement (148) is of the same order of magnitude as the maximal displacement during the passage of the gravitational wave train, if its source is a binary coalescence. In this case the average values over a suitable time period of the kinetic energy T of the relative motion and of the potential energy U of the binary bound stellar system obey, according to the *Virial theorem*,

$$2\,\langle T\rangle + \langle U\rangle = 0\,,$$

and, in any case, according to the energy theorem,

$$\langle T\rangle + \langle U\rangle = E_M \leq 0\,,$$

where E_M denotes the total mechanical energy of the bound astronomical bodies. We can think of the binary coalescence to have resulted by radiative capture from an initial system where the bodies are far away,

$$E_M = 0 \qquad \text{(initially)}.$$

So if we denote by E_R the total energy radiated to infinity up to a given time, by conservation of energy

$$E_M + E_R = 0\,;$$

(note that the right hand side is the initial value). It follows that

$$E_R = -E_M = \langle T\rangle\,.$$

Now on one hand

$$E_R \sim \int_{S^2} F \sim |\Sigma^+ - \Sigma^-| \, ,$$

and on the other

$$\sup_u |\Sigma^{(u)} - \Sigma^-| \sim \frac{\mathrm{d}^2}{\mathrm{d}t^2} Q \sim M\dot{X}^2 \sim \langle T \rangle \, ,$$

thus the permanent and maximal displacements are comparable:

$$|\Sigma^+ - \Sigma^-| \sim \sup_u |\Sigma^{(u)} - \Sigma^-| \, .$$

We expect that the displacement (148) builds up over the time scale over which the energy is radiated.

Remark 11. The appearance of trace-free symmetric 2×2-matrices in (146), (147), and (148) can be traced back to the *symbol* of the Einstein equations (5): This is the part containing the highest derivatives (principle part) with $\partial_\mu \partial_\nu g_{\alpha\beta}$ replaced by $\xi_\mu \xi_\nu \dot{g}_{\alpha\beta}$,

$$(\sigma_\xi \cdot \dot{g})_{\mu\nu} = \frac{1}{2}(g^{-1})^{\alpha\beta} \left(\xi_\mu \xi_\alpha \dot{g}_{\beta\nu} + \xi_\nu \xi_\alpha \dot{g}_{\beta\mu} - \xi_\mu \xi_\nu \dot{g}_{\alpha\beta} - \xi_\alpha \xi_\beta \dot{g}_{\mu\nu} \right) \, ,$$

(where ξ_μ are the components of a covector ξ and $\dot{g}_{\alpha\beta}$ are the components of a possible variation of g). After the removal of a degeneracy due to the general covariance of the equations, the null space of σ_ξ is found to be non-trivial for any null covector $\xi \neq 0$ and can be identified with the space of trace-free symmetric 2×2 matrices, (see [Ch-I]). This is the space of dynamical degrees of freedom of the gravitational field at a point.

References

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