An Introduction to Proof Theory

Class 3: Hypersequents for Modal Logics

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The modal logic of equivalence relations.

The Modal Logic S5

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Equivalently, it is the modal logic of *universal* relations.

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A model is a pair $\langle W, v \rangle$.

$$v_w(\Box A) = 1 \text{ iff for every } u, v_u(A) = 1$$

$$v_w(\Diamond A) = 1 \text{ iff for some } u, v_u(A) = 1$$

How can we simplify hypersequents for s5?

$$\frac{\mathcal{H}[X \vdash Y \frown X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \frown X' \vdash Y']}\,[\Box L]$$

$$\frac{\mathcal{H}[X \vdash Y \frown \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\,[\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \frown A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\,[\Diamond L]$$

$$\frac{\mathcal{H}[X \vdash Y \frown X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \frown X' \vdash Y']}\,[\Diamond R]$$

How can we simplify hypersequents for S5?

$$\frac{\mathcal{H}[X \vdash Y \rightharpoonup X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \rightharpoonup X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \rightharpoonup\ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \rightharpoonup A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \rightharpoonup X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \rightharpoonup X' \vdash Y']}\ [\Diamond R]$$

Eliminate the arrows!

A *flat hypersequent* is a non-empty multiset of sequents.

$$X_1 \vdash Y_1 \mid X_2 \vdash Y_2 \mid \cdots \mid X_n \vdash Y_n$$

FLAT HYPERSEQUENTS

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \frown X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \frown X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \frown \ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \frown A \vdash \ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \frown X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \frown X' \vdash Y']}\ [\Diamond R]$$

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$$\frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\ [\Diamond R]$$

$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ is a hypersequent in which $X \vdash Y$ and $X' \vdash Y'$ are components.

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \mid \ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\,[\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash \ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\,[\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\,[\Diamond R]$$

$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ is a hypersequent
in which $X \vdash Y$ and $X' \vdash Y'$ are components.

There is *subtlety* here—concerning reflexivity.

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$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\,[\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\,[\Diamond R]$$

$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ is a hypersequent
in which $X \vdash Y$ and $X' \vdash Y'$ are components.

There is *subtlety* here—concerning reflexivity.

In $\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ the $X \vdash Y$ and $X' \vdash Y'$ can be *the same*.

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \mid X' \vdash Y']}\ [eK]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \mid X' \vdash Y']}\ [eK]$$

$$\mathcal{H}[X, A \vdash A, Y] \qquad [axK]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\ [iWR]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\ [iWR]$$

$$\frac{\mathcal{H}[X \vdash Y \mid X' \vdash Y']}{\mathcal{H}[X, X' \vdash Y, Y']}\ [eWo]$$

Forms of *Cut*

$$\frac{X \vdash A, Y \mid \mathcal{H} \qquad X, A \vdash Y \mid \mathcal{H}}{X \vdash Y \mid \mathcal{H}}\ [aCut]$$

$$\frac{X \vdash A, Y \mid \mathcal{H} \qquad X', A \vdash Y' \mid \mathcal{H}'}{X, X' \vdash Y, Y' \mid \mathcal{H} \mid \mathcal{H}'}\ [mCut]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \;|\; \vdash A}\,[\Box L]}{\Box A, \Box B \vdash \;|\; \vdash A}\,[K] \qquad \dfrac{\dfrac{B \vdash B}{\Box B \vdash \;|\; \vdash B}\,[\Box L]}{\Box A, \Box B \vdash \;|\; \vdash B}\,[K]}{\Box A, \Box B \vdash \;|\; \vdash A \wedge B}\,[\wedge R]}{\Box A, \Box B \vdash \Box(A \wedge B)}\,[\Box R]}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}\,[\wedge R]$$

More Example Derivations

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \ \mid\ \vdash A}[\Box L]}{\Box A \vdash \ \mid\ \vdash \Box A}[\Box R]}{\Box A \vdash \Box\Box A}[\Box R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}[\neg L]}{\Box\neg A \vdash \ \mid\ A \vdash}[\Box L]}{\vdash \neg\Box\neg A \ \mid\ A \vdash}[\neg R]}{A \vdash \Box\neg\Box\neg A}[\Box R]$$

Modifying the Hypersequent Rules for S5

$$\frac{\mathcal{H}[X, \Box A \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash \Box A, Y \ \mid \ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \ \mid \ A \vdash]}{\mathcal{H}[X, \Diamond A \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash \Diamond A, Y \ \mid \ X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \ \mid \ X' \vdash Y']}\ [\Diamond R]$$

Height Preserving Admissibility

With these modified rules,
internal and external *weakening*,
and internal and external *contraction*,
are height-preserving admissible.

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The von Plato–Negri cut elimination argument
works straightforwardly for this system.
(See Poggiolesi 2008.)

(*m*)*Cut* Elimination: the $\Box$ Case

$$
\dfrac{\dfrac{\dfrac{\delta_1}{X \vdash Y \mid\ \vdash A \mid \mathcal{H}}}{X \vdash \Box A, Y \mid \mathcal{H}}[\Box R] \qquad \dfrac{\dfrac{\delta_1}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X', \Box A \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H}'}[\Box L]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}[mCut]
$$

$(m)Cut$ Elimination: the $\Box$ Case

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid\ \vdash A \mid \mathcal{H}}}{X \vdash \Box A, Y \mid \mathcal{H}}\,[\Box R] \qquad \dfrac{\dfrac{\delta_l}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X', \Box A \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H}'}\,[\Box L]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]$$

simplifies to

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid\ \vdash A \mid \mathcal{H}} \qquad \dfrac{\delta_r}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X \vdash Y \mid X' \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[eW]$$

Hypersequent Validity

$$X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$$

holds in $\mathfrak{M}$ iff there are no worlds w_i where
each element of X_i is true at w_i
and each element of Y_i is false at w_i.

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and each element of Y_i is false at w_i.

Equivalent *formula*:

$$\neg(\Diamond(\bigwedge X_1 \wedge \neg \bigvee Y_1) \wedge \cdots \wedge \Diamond(\bigwedge X_n \wedge \neg \bigvee Y_n))$$

$$X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$$

holds in $\mathfrak{M}$ iff there are no worlds w_i where
each element of X_i is true at w_i
and each element of Y_i is false at w_i.

Equivalent *formula*:

$$\neg(\Diamond(\bigwedge X_1 \wedge \neg \bigvee Y_1) \wedge \cdots \wedge \Diamond(\bigwedge X_n \wedge \neg \bigvee Y_n))$$

$$\Box(\bigwedge X_1 \supset \bigvee Y_1) \vee \cdots \vee \Box(\bigwedge X_n \supset \bigvee Y_n)$$

Features of this Proof System

Soundness and Completeness

Separation

Decision Procedure

Easy Extension

TWO DIMENSIONAL MODAL LOGIC

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The modal logic of *universal* relations with a distinguished world $w_@$.

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$$v_w(\Box A) = 1 \text{ iff for every } u, v_u(A) = 1$$

$$v_w(\Diamond A) = 1 \text{ iff for some } u, v_u(A) = 1$$

$$v_w(@A) = 1 \text{ iff } v_{w_@}(A) = 1$$

Hypersequents with @

$$X_1 \vdash Y_1 \ \mid \ \cdots \ \mid \ X_n \vdash Y_n$$

$$X_1 \vdash_{@} Y_1 \ \mid \ \cdots \ \mid \ X_n \vdash Y_n$$

Multisets of sequents where one (at most) is tagged with the label '@'.

Hypersequents with @

$$X_1 \vdash Y_1 \;\mid\; \cdots \;\mid\; X_n \vdash Y_n$$

$$X_1 \vdash_{@} Y_1 \;\mid\; \cdots \;\mid\; X_n \vdash Y_n$$

Multisets of sequents where one (at most) is tagged with the label '@'.

When you take the union of two hypersequents with @,
the @-sequents in the parent hypersequents are *merged*.

$$(X_1 \vdash_{@} Y_1 \;\mid\; X_2 \vdash Y_2) \;\mid\; (X_1' \vdash_{@} Y_1' \;\mid\; X_2' \vdash Y_2') =$$

$$X_1, X_1' \vdash_{@} Y_1, Y_1' \;\mid\; X_2 \vdash Y_2 \;\mid\; X_2' \vdash Y_2'$$

Rules for the @ operator

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash_{@} Y']}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_{@} Y']} \; [@L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash_{@} A, Y']}{\mathcal{H}[X \vdash @A, Y \mid X' \vdash_{@} Y']} \; [@R]$$

@-Hypersequent Notation

$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ — a hypersequent with components $X \vdash Y$ and $X' \vdash Y'$, which may or may not be identical.

$\mathcal{H}[X \vdash Y]$ — a hypersequent with a component $X \vdash Y$, which may or may not be tagged with '@'.

$\mathcal{H}[X \vdash_! Y]$ — a hypersequent with a component $X \vdash Y$, which is *not* tagged with '@.'

$\mathcal{H}[X \vdash_@ Y]$ — a hypersequent with a component $X \vdash_@ Y$, if X or Y are non-empty.

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']} [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \mid \vdash_! A]}{\mathcal{H}[X \vdash \Box A, Y]} [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash_!]}{\mathcal{H}[\Diamond A, X \vdash Y]} [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']} [\Diamond R]$$

Here, can't tag the $A \vdash$ component of $[\Diamond L]$
and the $\vdash A$ component of $[\Box R]$ with @.

(If we tag it, the premise is not general enough.)
We have $\vdash_@ p \supset @p$, but not $\vdash_@ \Box(p \supset @p)$.

The proviso on $X \vdash_@ Y$ ...

... means that the inference step

$$\frac{A \vdash_@}{@A \vdash}\ [@L]$$

is indeed an instance of $[@L]$ as it is specified.

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash_@ Y']}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_@ Y']}\ [@L]$$

Example Derivations

$$
\dfrac{\dfrac{\dfrac{p \vdash_{@} p \;\mid\; \vdash}{p \vdash_{@} \;\mid\; \vdash @p}\,[@R]}{p \vdash_{@} \Box @p}\,[\Box R]}{\vdash_{@} p \supset \Box @p}\,[\supset R]
$$

Example Derivations

$$\dfrac{\dfrac{\dfrac{p \vdash_{@} p \mid\ \vdash}{p \vdash_{@}\ \mid\ \vdash @p}[@R]}{p \vdash_{@} \Box @p}[\Box R]}{\vdash_{@} p \supset \Box @p}[\supset R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p \mid\ \vdash}{p \vdash_{@}\ \mid\ \vdash @p}[@R]}{p \vdash_{@} \Box @p}[\Box R]}{\vdash_{@} p \supset \Box @p}[\supset R]}{\vdash @(p \supset \Box @p)}[@R]}{\vdash \Box @(p \supset \Box @p)}[\Box R]$$

(m)*Cut* Elimination is unscathed

$$
\dfrac{
\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_@ A, Y' \mid \mathcal{H}}}{X \vdash @A, Y \mid X' \vdash_@ Y' \mid \mathcal{H}}\,[@R]
\qquad
\dfrac{\dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_@ Y''' \mid \mathcal{H}'}}{X'', @A \vdash Y'' \mid X''' \vdash_@ Y''' \mid \mathcal{H}'}\,[@L]
}{
X, X'' \vdash Y, Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'
}\,[mCut]
$$

(m)*Cut* Elimination is unscathed

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_@ A, Y' \mid \mathcal{H}}}{X \vdash @A, Y \mid X' \vdash_@ Y' \mid \mathcal{H}}\,[@R] \quad \dfrac{\dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_@ Y''' \mid \mathcal{H}'}}{X'', @A \vdash Y'' \mid X''' \vdash_@ Y''' \mid \mathcal{H}'}\,[@L]}{X, X'' \vdash Y, Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]$$

simplifies to

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_@ A, Y' \mid \mathcal{H}} \quad \dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_@ Y''' \mid \mathcal{H}'}}{X \vdash Y \mid X'' \vdash Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]}{X, X'' \vdash Y, Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}\,[eW]$$

A 2D model is a pair $\langle W, v \rangle$.

$$v_{w,w'}(\Box A) = 1 \text{ iff for every } u; v_{u,w'}(A) = 1$$

$$v_{w,w'}(\Diamond A) = 1 \text{ iff for some } u; v_{u,w'}(A) = 1$$

$$v_{w,w'}(@A) = 1 \text{ iff } v_{w',w'}(A) = 1$$

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$$v_{w,w'}(\Diamond A) = 1 \text{ iff for some } u;\, v_{u,w'}(A) = 1$$

$$v_{w,w'}(@A) = 1 \text{ iff } v_{w',w'}(A) = 1$$

$$v_{w,w'}(\mathsf{F}A) = 1 \text{ iff for every } u,\, v_{w,u}(A) = 1$$

Martin Davies & Lloyd Humberstone

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1						
w_2 w_3 $\vdots$ w_n $\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$	A	A	$\cdots$	A	$\cdots$
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A	A	A	$\cdots$	A	$\cdots$
w_2 w_3 $\vdots$ w_n $\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$	A	A	$\cdots$	A	$\cdots$
w_2 w_3 $\vdots$ w_n $\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$	A	A	$\cdots$	A	$\cdots$
w_2	$@B$					
w_3	$@B$					
$\vdots$	$\vdots$					
w_n	$@B$					
$\vdots$	$\vdots$					

□ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$ $@B$	A	A	$\cdots$	A	$\cdots$
w_2	$@B$					
w_3	$@B$					
$\vdots$	$\vdots$					
w_n	$@B$					
$\vdots$	$\vdots$					

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
	A	A	A	$\cdots$	A	$\cdots$
	$\mathrm{F@B}$					
	$\mathrm{@B},\ \mathrm{B}$					
w_2	$\mathrm{@B}$					
w_3	$\mathrm{@B}$					
$\vdots$	$\vdots$					
w_n	$\mathrm{@B}$					
$\vdots$	$\vdots$					

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
	A	A	A	$\cdots$	A	$\cdots$
	$F@B$					
	$@B,\ B$					
w_2	$@B$	B				
w_3	$@B$		B			
$\vdots$	$\vdots$			$\ddots$		
w_n	$@B$				B	
$\vdots$	$\vdots$					$\ddots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\ddots$		
w_n					B	
$\vdots$						$\ddots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	 A	 A	 $\cdots$	 A	 $\cdots$
w_2		B				
w_3			B			
$\vdots$				$\ddots$		
w_n					B	
$\vdots$						$\ddots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\ddots$		
w_n					B	
$\vdots$						$\ddots$

□ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\ddots$		
w_n					B	
$\vdots$						$\ddots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
	A	A	A	$\cdots$	A	$\cdots$
	$[K]B$					
	B					
w_2		B				
w_3			B			
$\vdots$				$\ddots$		
w_n					B	
$\vdots$						$\ddots$

Different Alternatives

$$\Box p \vdash \quad | \quad \vdash p$$

$$[K]p \vdash \quad || \quad \vdash_{@} p$$

In fact, we will have the following sort of derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p}{[\mathsf{K}]p \vdash \ \ \| \ \ \vdash_{@} p}}{[\mathsf{K}]p \vdash \ \ | \ \ \vdash [\mathsf{K}]p}\,[[\mathsf{K}]R]}{[\mathsf{K}]p \vdash \Box[\mathsf{K}]p}\,[\Box R]}{\vdash [\mathsf{K}]p \supset \Box[\mathsf{K}]p}\,[\supset R]
$$

$$
\begin{array}{ccccccccc}
X_1^1 \vdash_@ Y_1^1 & | & X_2^1 \vdash Y_2^1 & | & \cdots & | & X_{m_1}^1 \vdash Y_{m_1}^1 & \| \\
X_1^2 \vdash_@ Y_1^2 & | & X_2^2 \vdash Y_2^2 & | & \cdots & | & X_{m_2}^2 \vdash Y_{m_2}^2 & \| \\
\vdots & & \vdots & & & & \vdots & \\
X_1^n \vdash_@ Y_1^n & | & X_2^n \vdash Y_2^n & | & \cdots & | & X_{m_n}^n \vdash Y_{m_n}^n &
\end{array}
$$

2D Hypersequent Notation

$$\mathcal{H}[X \vdash Y \;\mid\; X' \vdash Y']$$

$$\mathcal{H}[X \vdash Y \;\|\; X' \vdash Y']$$

2D Hypersequent Rules

$$\frac{\mathcal{H}[X \vdash Y \;\parallel\; X', A \vdash_{@} Y']}{\mathcal{H}[X, [\mathrm{K}]A \vdash Y \;\parallel\; X' \vdash_{@} Y']}\ [\mathrm{APK}\ L]$$

$$\frac{\mathcal{H}[\ \vdash_{@} A \;\parallel\; X \vdash Y]}{\mathcal{H}[X \vdash [\mathrm{K}]A, Y]}\ [\mathrm{APK}\ R]$$

Example Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p}{p \vdash_{@} @p}\,[@R]}{\vdash_{@} p \supset @p}\,[\supset R]}{\vdash [\mathrm{K}](p \supset @p)}\,[[\mathrm{K}]R]}{\vdash \Box[\mathrm{K}](p \supset @p)}\,[\Box R]
$$

Cut Elimination is standard

$$\dfrac{\dfrac{\dfrac{\delta_1}{\mathcal{H}[\ \vdash_@ A \ \| \ X \vdash Y \ \| \ X' \vdash_@ Y']}}{\mathcal{H}[X \vdash [K]A, Y \ \| \ X' \vdash_@ Y']}\,[\text{APK } R] \quad \dfrac{\dfrac{\delta_2}{\mathcal{H}[X \vdash Y \ \| \ X', A \vdash_@ Y']}}{\mathcal{H}[X, [K]A \vdash Y \ \| \ X' \vdash_@ Y']}\,[\text{APK } L]}{\mathcal{H}[X \vdash Y \ \| \ X' \vdash_@ Y']}\,[aCut]$$

$$\dfrac{\dfrac{\dfrac{\delta_1}{\mathcal{H}[\ \vdash_@ A \ \| \ X \vdash Y \ \| \ X' \vdash_@ Y']} \quad \dfrac{\delta_2}{\mathcal{H}[X \vdash Y \ \| \ X', A \vdash_@ Y']}}{\mathcal{H}[X \vdash Y \ \| \ X' \vdash_@ Y' \ \| \ X' \vdash_@ Y']}\,[aCut]}{\mathcal{H}[X \vdash Y \ \| \ X' \vdash_@ Y']}\,[eW]$$

Proof Search for invalid sequents generates models

As in the classical case, we can use proof search on invalid sequents to generate models

Rather than a single position, $[X : Y]$, one uses a *modal position* made up of a multiset of positions, with one or more component positions designated as actual.

$$
\begin{array}{ccccccccc}
[X^1_1 : Y^1_1]_@ & | & [X^1_2 : Y^1_2] & | & \cdots & | & [X^1_{m_1} : Y^1_{m_1}] & \| \\
[X^2_1 : Y^2_1]_@ & | & [X^2_2 : Y^2_2] & | & \cdots & | & [X^2_{m_2} : Y^2_{m_2}] & \| \\
\vdots & & \vdots & & & & \vdots & \\
[X^n_1 : Y^n_1]_@ & | & [X^n_2 : Y^n_2] & | & \cdots & | & [X^n_{m_n} : Y^n_{m_n}] &
\end{array}
$$

What we've done

We've seen how the hypersequent calculus is not only a general technique for giving a sequent style proof theory for a range of propositional modal logics, but it can also be *tailored* to give simple proof systems for specific modal logics, with separable rules, and structural features neatly matched to the frame conditions for those logics.

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THANK YOU!

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https://consequently.org/class/2016/PTPLA-NASSLLI/