An Introduction to Proof Theory

Class 3: Beyond Sequents

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To introduce *proof theory*, with a focus on its applications in philosophy, linguistics and computer science.

Introduce extensions of sequent systems to naturally deal with modal logics.
Explore the behaviour of hypersequent systems for modal logics, including two dimensional modal logic with more than one modal operator.

Basic Modal Logic

Modal Sequent Systems

Display Logic

Labelled Sequents

Tree Hypersequents

BASIC MODAL LOGIC

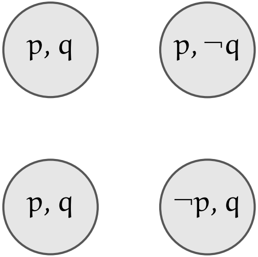
Possibility and Necessity

Modal logic adds propositional logic the notions of *possibility* and *necessity*.

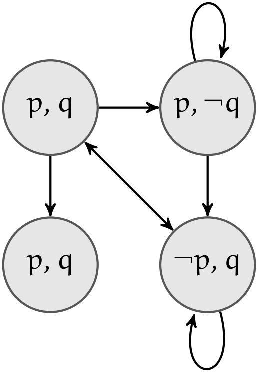
Add to the language of propositional logic the '$\Box$' and '$\Diamond$.'

- If A is a formula, so are $\Box A$ and $\Diamond A$.

Example Interpretation



Example Interpretation



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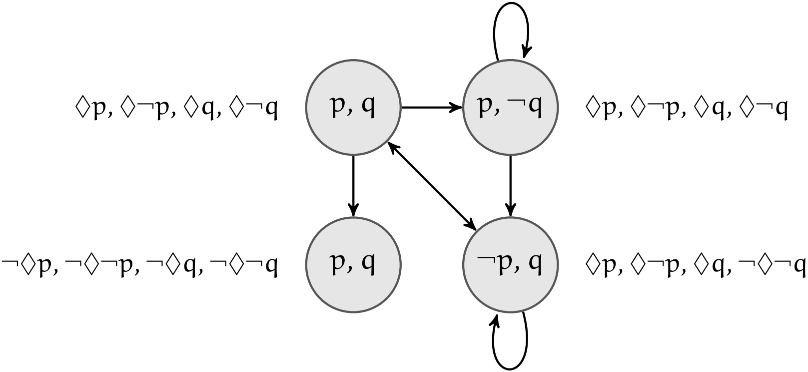

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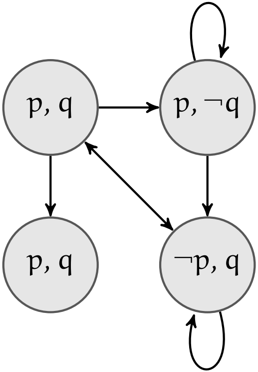

$\Diamond p, \Diamond\neg p, \Diamond q, \Diamond\neg q$

$\Diamond(p \wedge q)$

$\Diamond p, \Diamond\neg p, \Diamond q, \Diamond\neg q$

$\neg\Diamond(p \wedge q)$

$\neg\Diamond p, \neg\Diamond\neg p, \neg\Diamond q, \neg\Diamond\neg q$

$\neg\Diamond(p \wedge q)$

$\Diamond p, \Diamond\neg p, \Diamond q, \neg\Diamond\neg q$

$\Diamond(p \wedge q)$

Modal Logic: Interpretations

An *interpretation* for the language is a triple: $\langle W, R, v\rangle$.

W is a non-empty set of *states* (or *possible worlds*).

R is a two-place relation on W, of *relative possibility*. uRw means that from the point of view of u, w is possible.

Finally, v assigns a truth value to a propositional parameter *at a state*.

That is, for each world w and propositional parameter p, we will have either $v_w(p) = 1$ (if p is "true at w") or $v_w(p) = 0$ (if p is "false at w").

We keep the rules for the classical connectives, with state subscripts on v:

- $v_w(\neg A) = 1$ if and only if $v_w(A) = 0$.
- $v_w(A \wedge B) = 1$ if and only if $v_w(A) = 1$ and $v_w(B) = 1$.
- $v_w(A \vee B) = 1$ if and only if $v_w(A) = 1$ or $v_w(B) = 1$.
- $v_w(A \supset B) = 1$ if and only if $v_w(A) = 0$ or $v_w(B) = 1$.

No novelty there.

Interpreting the Language

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- $v_w(A \supset B) = 1$ if and only if $v_w(A) = 0$ or $v_w(B) = 1$.

No novelty there.

The innovation is found with $\Box$ and $\Diamond$:

- $v_w(\Box A) = 1$ if and only if $v_u(A) = 1$ for each u where wRu.
- $v_w(\Diamond A) = 1$ if and only if $v_u(A) = 1$ for some u where wRu.

Interpretations can be used to define validity, as with classical propositional logic.

Modal Validity

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The argument from X to Y is *valid* (written '$X \vdash Y$' as before) if and only if for every interpretation $\langle W, R, v \rangle$ for any state $w \in W$, if $v_w(B) = 1$ for each $B \in X$ then for some $C \in Y$, $v_w(C) = 1$ too.

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... or equivalently, there is no state $w \in W$ at which every member of X is true and every member of Y is false.

Some Basic Validity Facts

$$\frac{\vdash A}{\vdash \Box A}$$

Some Basic Validity Facts

$$\frac{\vdash A}{\vdash \Box A} \qquad \frac{A \vdash}{\Diamond A \vdash}$$

Some Basic Validity Facts

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$$\frac{A \vdash}{\Diamond A \vdash}$$

$$\frac{A \vdash B}{\Box A \vdash \Box B}$$

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$$\frac{A \vdash}{\Diamond A \vdash}$$

$$\frac{A \vdash B}{\Box A \vdash \Box B}$$

$$\frac{A \vdash B}{\Diamond A \vdash \Diamond B}$$

$$\frac{X, \Box A, \Box B \vdash Y}{X, \Box(A \wedge B) \vdash Y}$$

Some Basic Validity Facts

$$\frac{\vdash A}{\vdash \Box A}$$

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$$\frac{A \vdash B}{\Box A \vdash \Box B}$$

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$$\begin{array}{c} X, \Box A, \Box B \vdash Y \\ \hline\hline X, \Box(A \wedge B) \vdash Y \end{array}$$

$$\begin{array}{c} X \vdash \Diamond A, \Diamond B, Y \\ \hline\hline X \vdash \Diamond(A \vee B), Y \end{array}$$

Some Basic Validity Facts

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$$\frac{A \vdash B}{\Box A \vdash \Box B} \qquad \frac{A \vdash B}{\Diamond A \vdash \Diamond B}$$

$$\frac{X, \Box A, \Box B \vdash Y}{X, \Box(A \wedge B) \vdash Y} \qquad \frac{X \vdash \Diamond A, \Diamond B, Y}{X \vdash \Diamond(A \vee B), Y}$$

$$\frac{X, \Box A \vdash Y}{X, \neg\Diamond\neg A \vdash Y}$$

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None of these are much like good L/R rules for $\Box$ or $\Diamond$.

Restrictions on the accessibility relation lead to properties for $\Box$ and $\Diamond$.

CONDITION		PROPERTY
reflexivity	wRw	$\Box A \vdash A \quad A \vdash \Diamond A.$

Moving Beyond Basic Modal Logic

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symmetry	$wRv \supset vRw$	$A \vdash \Box\Diamond A$	$\Diamond\Box A \vdash A.$

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directedness	$(\exists v)wRv$	$\Box\bot \vdash$	$\vdash \Diamond\top$

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$\vdots$		$\vdots$	

K: *all* models T: *reflexive* models S4: *reflexive transitive* models
S5: *reflexive symmetric transitive* models.

Brief aside: Intuitionistic models

We can use Kripke models as the model theory for intuitionistic logic, where the accessibility relation of the models is reflexive and transitive.

- $v_w(A \land B) = 1$ if and only if $v_w(A) = 1$ and $v_w(B) = 1$.
- $v_w(A \lor B) = 1$ if and only if $v_w(A) = 1$ or $v_w(B) = 1$.

Conjunction and disjunction are standard.

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- $v_w(A \vee B) = 1$ if and only if $v_w(A) = 1$ or $v_w(B) = 1$.

Conjunction and disjunction are standard.

Negation and the conditional receive different truth conditions.

- $v_w(\neg A) = 1$ if and only if $v_u(A) = 0$ for each u where wRu.
- $v_w(A \to B) = 1$ if and only if $v_u(A) = 0$ or $v_u(B) = 1$, for each u where wRu.

Finally, we also require a Heredity Condition: for atoms p, if $v_w(p) = 1$ and wRu, then $v_u(p) = 1$.

MODAL SEQUENT SYSTEMS

What could L/R rules for $\Box$ and $\Diamond$ look like?

$$\frac{??? \vdash ???}{X \vdash \Box A, Y}$$

What could L/R rules for $\Box$ and $\Diamond$ look like?

$$\frac{X \vdash A}{\Box X \vdash \Box A}$$

What could L/R rules for $\Box$ and $\Diamond$ look like?

$$\frac{\Box X \vdash A}{\Box X \vdash \Box A}$$

What could L/R rules for $\Box$ and $\Diamond$ look like?

$$\frac{\Box X \vdash A, \Diamond Y}{\Box X \vdash \Box A, \Diamond Y}\ [\Box R]$$

What could L/R rules for $\Box$ and $\Diamond$ look like?

$$\frac{X, A \vdash Y}{X, \Box A \vdash Y}\ [\Box L] \qquad \frac{\Box X \vdash A, \Diamond Y}{\Box X \vdash \Box A, \Diamond Y}\ [\Box R]$$

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$$\frac{\Box X, A \vdash \Diamond Y}{\Box X, \Diamond A \vdash \Diamond Y}\,[\Diamond L] \qquad \frac{X \vdash A, Y}{X \vdash \Diamond A, Y}\,[\Diamond R]$$

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These rules characterise the modal logic S4.

Example Derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\,[\wedge R]}{\Box A, B \vdash A \wedge B}\,[\Box L]}{\Box A, \Box B \vdash A \wedge B}\,[\Box L]}{\Box A, \Box B \vdash \Box(A \wedge B)}\,[\Box R]}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}\,[\wedge L]$$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash A}\,[\Box L]}{\Box A \vdash \Box A}\,[\Box R]}{\Box A \vdash \Box\Box A}\,[\Box R]$$

What about S5?

$$\frac{\Box X \vdash A, \Box Y}{\Box X \vdash \Box A, \Box Y}\ [\Box R']$$

$$\frac{\Diamond X, A \vdash \Diamond Y}{\Diamond X, \Diamond A \vdash \Diamond Y}\ [\Diamond L']$$

What about S5?

$$\frac{\Box X \vdash A, \Box Y}{\Box X \vdash \Box A, \Box Y}\ [\Box R']$$

$$\frac{\Diamond X, A \vdash \Diamond Y}{\Diamond X, \Diamond A \vdash \Diamond Y}\ [\Diamond L']$$

$$\dfrac{\dfrac{\dfrac{\Box p \vdash \Box p}{\vdash \Box p, \neg\Box p}\ [\neg R]}{\vdash \Box p, \Box\neg\Box p}\ [\Box R'] \qquad \dfrac{p \vdash p}{\Box p \vdash p}\ [\Box L]}{\vdash p, \Box\neg\Box p}\ [Cut]$$

What about S5?

$$\frac{\Box X \vdash A, \Box Y}{\Box X \vdash \Box A, \Box Y}\,[\Box R']$$

$$\frac{\Diamond X, A \vdash \Diamond Y}{\Diamond X, \Diamond A \vdash \Diamond Y}\,[\Diamond L']$$

$$\dfrac{\dfrac{\dfrac{\Box p \vdash \Box p}{\vdash \Box p, \neg\Box p}\,[\neg R]}{\vdash \Box p, \Box\neg\Box p}\,[\Box R'] \qquad \dfrac{p \vdash p}{\Box p \vdash p}\,[\Box L]}{\vdash p, \Box\neg\Box p}\,[Cut]$$

The sequent $\vdash p, \Box\neg\Box p$ has *no* cut-free proof.
(How could you apply a $\Box$ rule?)

Problems with these $\Box$ and $\Diamond$ rules

$$\frac{X, A \vdash Y}{X, \Box A \vdash Y}\,[\Box L] \qquad \frac{\Box X \vdash A, \Diamond Y}{\Box X \vdash \Box A, \Diamond Y}\,[\Box R] \qquad \frac{\Box X \vdash A, \Box Y}{\Box X \vdash \Box A, \Box Y}\,[\Box R']$$

$$\frac{\Box X, A \vdash \Diamond Y}{\Box X, \Diamond A \vdash \Diamond Y}\,[\Diamond L] \qquad \frac{\Diamond X, A \vdash \Diamond Y}{\Diamond X, \Diamond A \vdash \Diamond Y}\,[\Diamond L'] \qquad \frac{X \vdash A, Y}{X \vdash \Diamond A, Y}\,[\Diamond R]$$

Entanglement between $\Box$ and $\Diamond$.

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$$\frac{\Box X, A \vdash \Diamond Y}{\Box X, \Diamond A \vdash \Diamond Y}\,[\Diamond L] \qquad \frac{\Diamond X, A \vdash \Diamond Y}{\Diamond X, \Diamond A \vdash \Diamond Y}\,[\Diamond L'] \qquad \frac{X \vdash A, Y}{X \vdash \Diamond A, Y}\,[\Diamond R]$$

Entanglement between $\Box$ and $\Diamond$.

$\Box L$ and $\Diamond R$ are weak
— all the work is done by
the left $\Diamond$ rules and right $\Box$ rules.

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$$\frac{X, A \vdash Y}{X, \Box A \vdash Y}\,[\Box L] \qquad \frac{\Box X \vdash A, \Diamond Y}{\Box X \vdash \Box A, \Diamond Y}\,[\Box R] \qquad \frac{\Box X \vdash A, \Box Y}{\Box X \vdash \Box A, \Box Y}\,[\Box R']$$

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Entanglement between $\Box$ and $\Diamond$.

$\Box L$ and $\Diamond R$ are weak
— all the work is done by
the left $\Diamond$ rules and right $\Box$ rules.

Hard/impossible to generalise.

From Modal to *Temporal* Logic

- $v_w(\Box A) = 1$ if and only if $v_u(A) = 1$ for each u where wRu.
- $v_w(\Diamond A) = 1$ if and only if $v_u(A) = 1$ for some u where wRu.

- $v_w(\blacksquare A) = 1$ if and only if $v_u(A) = 1$ for each u where uRw.
- $v_w(\blacklozenge A) = 1$ if and only if $v_u(A) = 1$ for some u where uRw.

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$$\frac{A \vdash \Box B}{\blacklozenge A \vdash B} \qquad \frac{\Diamond A \vdash B}{A \vdash \blacksquare B}$$

Going Forward and Back in a Derivation

$$
\begin{prooftree}
\AxiomC{$\Box A, \Box B \vdash \Box A$}
\RightLabel{$[\wedge L]$}
\UnaryInfC{$\Box A \wedge \Box B \vdash \Box A$}
\RightLabel{$[\Box\blacklozenge]$}
\UnaryInfC{$\blacklozenge(\Box A \wedge \Box B) \vdash A$}
\AxiomC{$\Box A, \Box B \vdash \Box B$}
\RightLabel{$[\wedge L]$}
\UnaryInfC{$\Box A \wedge \Box B \vdash \Box B$}
\RightLabel{$[\Box\blacklozenge]$}
\UnaryInfC{$\blacklozenge(\Box A \wedge \Box B) \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$\blacklozenge(\Box A \wedge \Box B) \vdash A \wedge B$}
\RightLabel{$[\blacklozenge\Box]$}
\UnaryInfC{$\Box A \wedge \Box B \vdash \Box(A \wedge B)$}
\end{prooftree}
$$

Generalised Sequents

How do we establish $X \vdash \Box A, Y$?

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It should have *something* to do with some $X' \vdash A, Y'$

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We need to record state shifts in sequents.

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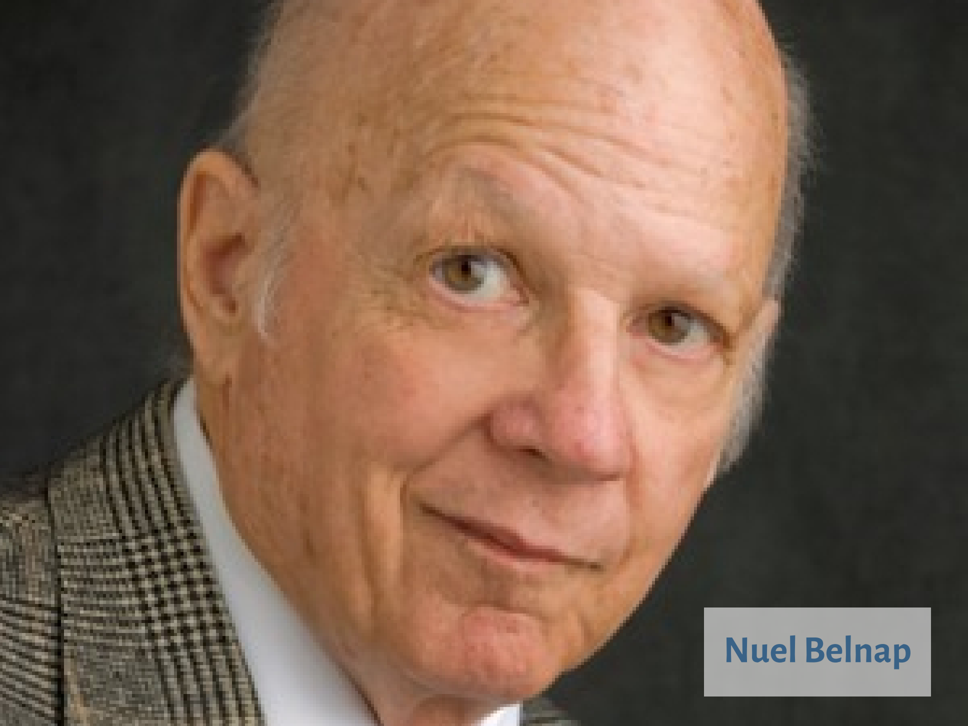
We need to record state shifts in sequents.

DISPLAY LOGIC ● LABELLED SEQUENTS ● TREE HYPERSEQUENTS

DISPLAY LOGIC



Nuel Belnap

Sequents

Sequents are of the form $X \vdash Y$, where X and Y are *structures*

Structures are built up out of formulas and the structural connetives $*$, $\bullet$ (both unary), and $\circ$ (binary)

For example, $*(p \circ q) \vdash \bullet(r \circ *s)$

Certain sequents are stipulated to be equivalent via *display equivalences*

$$X \vdash Y \circ Z \Longleftrightarrow X \circ *Y \vdash Z \Longleftrightarrow X \vdash Z \circ Y$$

$$X \vdash Y \Longleftrightarrow *Y \vdash *X \Longleftrightarrow X \vdash **Y$$

$$\bullet X \vdash Y \Longleftrightarrow X \vdash \bullet Y$$

Display equivalences

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$$X \vdash Y \Longleftrightarrow *Y \vdash *X \Longleftrightarrow X \vdash **Y$$

$$\bullet X \vdash Y \Longleftrightarrow X \vdash \bullet Y$$

(These rules ensure that $*$ acts like *negation*,
$\circ$ is *conjunctive* on the left and *disjunctive* on the right,
and $\bullet$ acts like a *necessity* on the right
and its *converse possibility* the left.)

Displaying

By means of the display equivalences, one can display a formula or structure on one side of the turnstile in isolation

This permits the left and right rules to deal with only the displayed formulas and structures

$$\frac{A \circ B \vdash X}{A \wedge B \vdash X}\,[\wedge L] \qquad\qquad \frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B}\,[\wedge R]$$

Generality

The connectives rules are formulated so that
each connective is paired with a structural connective

Different logical behaviour is obtained
by imposing different rules on the structural connectives

A single form of conjunction rule can be used for, say, classical conjunction
and relevant fusion, the difference coming out in the structural rules in force

To give rules for modal operators, you use the *modal structure*.

$$\frac{A \vdash Y}{\Box A \vdash \bullet Y}\ [\Box L] \qquad\qquad \frac{X \vdash \bullet B}{X \vdash \Box B}\ [\Box R]$$

Example Display Logic Derivation

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \bullet A$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A \circ \Box B \vdash \bullet A$}
\RightLabel{$[display]$}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box B \vdash \bullet B$}
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\UnaryInfC{$\Box A \circ \Box B \vdash \bullet B$}
\RightLabel{$[display]$}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$\bullet(\Box A \circ \Box B) \circ \bullet(\Box A \circ \Box B) \vdash A \wedge B$}
\RightLabel{$[W]$}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash A \wedge B$}
\RightLabel{$[display]$}
\UnaryInfC{$\Box A \circ \Box B \vdash \bullet(A \wedge B)$}
\RightLabel{$[\wedge L]$}
\UnaryInfC{$\Box A \wedge \Box B \vdash \bullet(A \wedge B)$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$\Box A \wedge \Box B \vdash \Box(A \wedge B)$}
\end{prooftree}
$$

Structural Rules

$$\frac{X \vdash \bullet Y}{X \vdash Y}\ [\mathit{refl}]$$

$$\cfrac{\cfrac{A \vdash A}{\Box A \vdash \bullet A}\ [\Box L]}{\Box A \vdash A}\ [\mathit{refl}]$$

Structural Rules

$$\frac{X \vdash \bullet Y}{X \vdash Y}\,[\mathit{refl}]$$

$$\frac{X \vdash \bullet Y}{X \vdash \bullet\bullet Y}\,[\mathit{trans}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \bullet A}\,[\Box L]}{\Box A \vdash \bullet\bullet A}\,[\mathit{trans}]}{\bullet\Box A \vdash \bullet A}\,[\mathit{display}]}{\bullet\Box A \vdash \Box A}\,[\Box R]}{\Box A \vdash \bullet\Box A}\,[\mathit{display}]}{\Box A \vdash \Box\Box A}\,[\Box R]$$

Structural Rules

$$\frac{X \vdash \bullet Y}{X \vdash Y}\ [\mathit{refl}]$$

$$\frac{X \vdash \bullet Y}{X \vdash \bullet\bullet Y}\ [\mathit{trans}]$$

$$\frac{X \vdash \bullet * Y}{X \vdash * \bullet Y}\ [\mathit{sym}]$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{*A \vdash *A}\ [\mathit{display}]}{\neg A \vdash *A}\ [\neg L]}{\Box\neg A \vdash \bullet * A}\ [\Box L]}{\Box\neg A \vdash * \bullet A}\ [\mathit{sym}]}{\bullet A \vdash * \Box\neg A}\ [\mathit{display}]}{\bullet A \vdash \neg\Box\neg A}\ [\neg R]}{A \vdash \bullet\neg\Box\neg A}\ [\mathit{display}]}{A \vdash \Box\neg\Box\neg A}\ [\Box R]$$

Structural Rules

$$\frac{X \vdash \bullet Y}{X \vdash Y}\ [\mathit{refl}]$$

$$\frac{X \vdash \bullet Y}{X \vdash \bullet\bullet Y}\ [\mathit{trans}]$$

$$\frac{X \vdash \bullet * Y}{X \vdash * \bullet Y}\ [\mathit{sym}]$$

Many more structural rules are possible.

Cut

Because formulas can always be displayed,
a simple form of *Cut* can be used for a range of logics

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,[\mathit{Cut}]$$

Eliminating Cut

The *Elimination Theorem* is proved via a general argument that depends on eight conditions on the rules.

If these conditions are satisfied, then it follows that *Cut* is admissible

This argument is due to *Haskell Curry* and *Nuel Belnap*.

The Structure of the Curry–Belnap Cut Elimination Proof

- It's a *Cut* elimination argument (it doesn't appeal to a Mix rule).
- It's an induction on *grade* (complexity of the *Cut* formula), as usual.
- To eliminate a *Cut* on a formula A, trace the *parametric* occurrences of a formula in the premises of the cut inference upward to where they first appear. Replace the cut at those instances (either with cuts on subformulas, or by weakening, or the cuts evaprate into identities) and then replay the substitution downward.

$$
\dfrac{X \vdash A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots A, A \cdots}}{\cdots A \cdots} \qquad \dfrac{\cdots A \cdots \qquad \dfrac{\dfrac{\vdots}{\cdots A, A \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{A \vdash Y}}{X \vdash Y}\,[Cut]
$$

The Crucial Step

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots A, A \cdots}}{\cdots A \cdots} \qquad \dfrac{\cdots A \cdots \qquad \dfrac{\dfrac{\vdots}{\cdots A, A \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{A \vdash Y}
$$

The Crucial Step

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots X, X \cdots}}{\cdots A \cdots} \qquad \dfrac{\cdots X \cdots \qquad \dfrac{\dfrac{\vdots}{\cdots X, X \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{\cdots A \cdots}}{A \vdash Y}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots X, X \cdots}}{\cdots X \cdots} \qquad \dfrac{\cdots X \cdots \qquad \dfrac{\dfrac{\vdots}{\cdots X, X \cdots}}{\cdots X \cdots}}{\cdots X \cdots}}{\cdots X \cdots}}{X \vdash Y}$$

The Eight Conditions

- C1: *Preservation of formulas.*
- C2: *Shape-alikeness of parameters.*
- C3: *Non-proliferation of parameters.*
- C4: *Position-alikeness of parameters.*
- C5: *Display of principal constituents.*
- C6: *Closure under substitution for consequent parameters.*
- C7: *Closure under substitution for antecedent parameters.*
- C8: *Eliminability of matching principal constituents.*

Cut Elimination: The $\Box$ Case

A cut on a principal $\Box A$ may be simplified into a cut on A.

$$\dfrac{\dfrac{X \vdash \bullet A}{X \vdash \Box A}\,[\Box R] \qquad \dfrac{A \vdash Y}{\Box A \vdash \bullet Y}\,[\Box L]}{X \vdash \bullet Y}\,[Cut]$$

Cut Elimination: The $\Box$ Case

A cut on a principal $\Box A$ may be simplified into a cut on A.

$$\dfrac{\dfrac{X \vdash \bullet A}{X \vdash \Box A}[\Box R] \qquad \dfrac{A \vdash Y}{\Box A \vdash \bullet Y}[\Box L]}{X \vdash \bullet Y}[Cut]$$

$$\dfrac{\dfrac{\dfrac{X \vdash \bullet A}{\bullet X \vdash A}[display] \qquad A \vdash Y}{\bullet X \vdash Y}[Cut]}{X \vdash \bullet Y}[display]$$

Virtues and Vices of Display Logic

	DISPLAY
Cut-free	$+$
Explicit	$+$
Systematic	$+$
Separation	$+$
Subformula	$+$
Nonredundant	$-$
Gentzen-plus	$-$

Virtues and Vices of Display Logic

	DISPLAY
Cut-free	+
Explicit	+
Systematic	+
Separation	+
Subformula	+
Nonredundant	—
Gentzen-plus	—

LABELLED SEQUENTS

Recall this derivation...

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{[$\Box L$]}
\UnaryInfC{$\Box A \vdash \bullet A$}
\RightLabel{[K]}
\UnaryInfC{$\Box A \circ \Box B \vdash \bullet A$}
\RightLabel{[$display$]}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{[$\Box L$]}
\UnaryInfC{$\Box B \vdash \bullet B$}
\RightLabel{[K]}
\UnaryInfC{$\Box A \circ \Box B \vdash \bullet B$}
\RightLabel{[$display$]}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash B$}
\RightLabel{[$\wedge R$]}
\BinaryInfC{$\bullet(\Box A \circ \Box B) \circ \bullet(\Box A \circ \Box B) \vdash A \wedge B$}
\RightLabel{[W]}
\UnaryInfC{$\bullet(\Box A \circ \Box B) \vdash A \wedge B$}
\RightLabel{[$display$]}
\UnaryInfC{$\Box A \circ \Box B \vdash \bullet(A \wedge B)$}
\RightLabel{[$\wedge L$]}
\UnaryInfC{$\Box A \wedge \Box B \vdash \bullet(A \wedge B)$}
\RightLabel{[$\Box R$]}
\UnaryInfC{$\Box A \wedge \Box B \vdash \Box(A \wedge B)$}
\end{prooftree}
$$

Here is another way to represent it

$$
\dfrac{\dfrac{\dfrac{\dfrac{v : A \vdash v : A}{wRv, w : \Box A \vdash v : A}[\Box L]}{wRv, w : \Box A, w : \Box B \vdash v : A}[K] \qquad \dfrac{\dfrac{v : B \vdash v : B}{wRv, w : \Box B \vdash v : B}[\Box L]}{wRv, w : \Box A, w : \Box B \vdash v : B}[K]}{\dfrac{wRv, w : \Box A, w : \Box B, wRv, w : \Box A, w : \Box B \vdash v : A \wedge B}{\dfrac{wRv, w : \Box A, w : \Box B \vdash A \wedge B}{wRv, w : \Box A \wedge \Box B \vdash v : A \wedge B}[\wedge L]}[W]}[\wedge R]}{w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)}[\Box R]
$$

Labelled Sequent Rules: Boolean Connectives

$$x : A \vdash x : A$$

(Plus weakening and contraction.)

$$\frac{x : A, x : B, X \vdash Y}{x : A \wedge B, X \vdash Y}\,[\wedge L] \qquad \frac{X \vdash x : A, Y \quad X \vdash x : B, Y}{X \vdash x : A \wedge B, Y}\,[\wedge R]$$

$$\frac{x : A, X \vdash Y \quad x : B, X \vdash Y}{x : A \vee B, X \vdash Y}\,[\vee L] \qquad \frac{X \vdash x : A, x : B, Y}{X \vdash x : A \vee B, Y}\,[\vee R]$$

$$\frac{X \vdash x : A, Y}{x : \neg A, X \vdash Y}\,[\neg L] \qquad \frac{x : A, X \vdash Y}{X \vdash x : \neg A, Y}\,[\neg R]$$

Labelled Sequent Rules: Modal Operators

$$\frac{x : A, X \vdash Y}{yRx, y : \Box A, X \vdash Y}\ [\Box L] \qquad \frac{xRy, X \vdash y : A, Y}{X \vdash x : \Box A, Y}\ [\Box R]$$

$$\frac{xRy, y : A, X \vdash Y}{x : \Diamond A, X \vdash Y}\ [\Diamond L] \qquad \frac{X \vdash x : A, Y}{yRx, X \vdash y : \Diamond A, Y}\ [\Diamond R]$$

In $\Box R$ and $\Diamond L$, the label y must not be present in X, Y or be identical to x.

Labelled Sequents

In these rules (except for weakenings) relational statements (xRy) are introduced only on the left of the sequent.

We may without loss of deductive power, restrict our attention to sequents in $X \vdash Y$ which relational statements appear only in X and not in Y.

Frame conditions

The 'cash value' of a labelled sequent $X \vdash Y$ on a Kripke model is found by replacing $x : A$ by $v_x(A) = 1$; X by its conjunction; Y by its disjunction; the $\vdash$ by a conditional; and universally quantifying over all world labels.

Frame conditions

The 'cash value' of a labelled sequent $X \vdash Y$ on a Kripke model is found by replacing $x : A$ by $v_x(A) = 1$; X by its conjunction; Y by its disjunction; the $\vdash$ by a conditional; and universally quantifying over all world labels.

$xRy, x : A \vdash y : B, x : C$ is valid on a model if and only if

$$(\forall x, y)((xRy \wedge v_x(A) = 1) \supset ((v_y(B) = 1) \vee v_x(C) = 1))$$

A systematic translation maps modal display derivations
into labelled modal derivations.

The translation simplifies the proof structure,
erasing display equivalences, which are
mapped to identical labelled sequents (*modulo* relabelling).

For details, see Poggiolesi and Restall
"Interpreting and Applying Proof Theory for Modal Logic" (2012).

Virtues and Vices

	DISPLAY	LABELLED
Cut-free	$+$	$+$
Explicit	$+$	$+$
Systematic	$+$	$+$
Separation	$+$	$+$
Subformula	$+$	$+-$
Nonredundant	$-$	$+-$
Gentzen-plus	$-$	$+-$

Virtues and Vices

	DISPLAY	LABELLED
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Gentzen-plus	$-$	$+-$

TREE HYPERSEQUENTS

Inspecting the translation

Display equivalent sequents correspond to *nearly* identical labelled sequents.

$$A \vdash \bullet B$$

$$\bullet A \vdash B$$

Inspecting the translation

Display equivalent sequents correspond to *nearly* identical labelled sequents.

$$A \vdash \bullet B \quad \Rightarrow \quad vRw, v : A \vdash w : B$$

$$\bullet A \vdash B$$

Inspecting the translation

Display equivalent sequents correspond to *nearly* identical labelled sequents.

$$A \vdash \bullet B \quad \Rightarrow \quad vRw, v : A \vdash w : B$$

$$\bullet A \vdash B \quad \Rightarrow \quad wRv, w : A \vdash v : B$$

Inspecting the translation

Display equivalent sequents correspond to *nearly* identical labelled sequents.

$$A \vdash \bullet B \quad \Rightarrow \quad vRw, v : A \vdash w : B$$

$$\bullet A \vdash B \quad \Rightarrow \quad wRv, w : A \vdash v : B$$

All we *care* about is that one world accesses the other. We have

$$A \vdash \longrightarrow \vdash B$$

The Recipe

Replace the labelled sequent $\mathcal{R}, X \vdash Y$ by a *directed graph* of sequents:

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The Recipe

Replace the labelled sequent $\mathcal{R}, X \vdash Y$ by a *directed graph* of sequents:

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Replace the labelled sequent $\mathcal{R}, X \vdash Y$ by a *directed graph* of sequents:

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Replace the labelled sequent $\mathcal{R}, X \vdash Y$ by a *directed graph* of sequents:

- There is one node for every label.
- Every node is a sequent.
- For every instance of $x : A$ in antecedent position, put A in the antecedent of the sequent at the node corresponding to x.
- For every instance of $x : A$ in consequent position, put A in the consequent of the sequent at the node corresponding to x.

Replace the labelled sequent $\mathcal{R}, X \vdash Y$ by a *directed graph* of sequents:

- There is one node for every label.
- Every node is a sequent.
- For every instance of $x : A$ in antecedent position, put A in the antecedent of the sequent at the node corresponding to x.
- For every instance of $x : A$ in consequent position, put A in the consequent of the sequent at the node corresponding to x.
- If $\mathcal{R}$ contains Rxy, then place an arc from x to y.

Three ways of presenting the one fact

- Display Sequent: $\bullet * (A \circ *\bullet B) \vdash *(D \circ E)$

Three ways of presenting the one fact

- Display Sequent: $\bullet * (A \circ *\bullet B) \vdash *(D \circ E)$
- Labelled Sequent: $vRw, uRv, u : B, w : D, w : E \vdash v : A$

Three ways of presenting the one fact

- Display Sequent: $\bullet * (A \circ *\bullet B) \vdash *(D \circ E)$
- Labelled Sequent: $vRw, uRv, u : B, w : D, w : E \vdash v : A$
- Delabelled Sequent: $B \vdash \longrightarrow D, E \vdash \longrightarrow \vdash A$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$v : A \vdash v : A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$wRv, w : \Box A \vdash v : A$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\AxiomC{$v : B \vdash v : B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$wRv, w : \Box A \vdash v : A$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\dfrac{
\dfrac{\dfrac{\color{red}{A \vdash A}}{wRv, w : \Box A \vdash v : A}\,[\Box L]}{wRv, w : \Box A, w : \Box B \vdash v : A}\,[K]
\qquad
\dfrac{\dfrac{v : B \vdash v : B}{wRv, w : \Box A \vdash v : A}\,[\Box L]}{wRv, w : \Box A, w : \Box B \vdash v : A}\,[K]
}{
\dfrac{\dfrac{wRv, w : \Box A, w : \Box B \vdash v : A \wedge B}{w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)}\,[\Box R]}{w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)}\,[\wedge R]
}\,[\wedge R]
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\color{red}{\Box A \vdash \longrightarrow \vdash A}$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\AxiomC{$v : B \vdash v : B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$wRv, w : \Box A \vdash v : A$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \longrightarrow \vdash A$}
\RightLabel{$[K]$}
\UnaryInfC{$\color{red}{\Box A, \Box B \vdash \longrightarrow \vdash A}$}
\AxiomC{$v : B \vdash v : B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$wRv, w : \Box A \vdash v : A$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \longrightarrow \vdash A$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A$}
\AxiomC{$\color{red}{B \vdash B}$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$wRv, w : \Box A \vdash v : A$}
\RightLabel{$[K]$}
\UnaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\dfrac{
\dfrac{\dfrac{A \vdash A}{\Box A \vdash \longrightarrow \vdash A}\,[\Box L]}{\Box A, \Box B \vdash \longrightarrow \vdash A}\,[K]
\qquad
\dfrac{\dfrac{B \vdash B}{\Box B \vdash \longrightarrow \vdash B}\,[\Box L]}{wRv, w : \Box A, w : \Box B \vdash v : A}\,[K]
}{
\dfrac{\dfrac{wRv, w : \Box A, w : \Box B \vdash v : A \wedge B}{w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)}\,[\Box R]}{w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)}\,[\wedge R]
}\,[\wedge R]
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \;\longrightarrow\; \vdash A$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \;\longrightarrow\; \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box B \vdash \;\longrightarrow\; \vdash B$}
\RightLabel{$[K]$}
\UnaryInfC{$\color{red}{\Box A, \Box B \vdash \;\longrightarrow\; \vdash B}$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$wRv, w : \Box A, w : \Box B \vdash v : A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \longrightarrow \vdash A$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box B \vdash \longrightarrow \vdash B$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{${\color{red}\Box A, \Box B \vdash \longrightarrow \vdash A \wedge B}$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$w : \Box A, w : \Box B \vdash w : \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{[$\Box L$]}
\UnaryInfC{$\Box A \vdash \longrightarrow \vdash A$}
\RightLabel{[K]}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{[$\Box L$]}
\UnaryInfC{$\Box B \vdash \longrightarrow \vdash B$}
\RightLabel{[K]}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash B$}
\RightLabel{[$\wedge R$]}
\BinaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A \wedge B$}
\RightLabel{[$\Box R$]}
\UnaryInfC{$\color{red}{\Box A, \Box B \vdash \Box(A \wedge B)}$}
\RightLabel{[$\wedge R$]}
\UnaryInfC{$w : \Box A \wedge \Box B \vdash w : \Box(A \wedge B)$}
\end{prooftree}
$$

An example delabelling

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box A \vdash \longrightarrow \vdash A$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box B \vdash \longrightarrow \vdash B$}
\RightLabel{$[K]$}
\UnaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash B$}
\RightLabel{$[\wedge R]$}
\BinaryInfC{$\Box A, \Box B \vdash \longrightarrow \vdash A \wedge B$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$\Box A, \Box B \vdash \Box(A \wedge B)$}
\RightLabel{$[\wedge R]$}
\UnaryInfC{$\color{red}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$x : A \vdash x : A$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$x : \neg A, x : A \vdash$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$Ryx, y : \Box\neg A, x : A \vdash$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$Ryx, x : A \vdash y : \neg\Box\neg A$}
\RightLabel{$[sym]$}
\UnaryInfC{$Rxy, x : A \vdash y : \neg\Box\neg A$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$x : A \vdash x : \Box\neg\Box\neg A$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$[\neg L]$}
\UnaryInfC{$\neg A, A \vdash$}
\RightLabel{$[\Box L]$}
\UnaryInfC{$\Box\neg A \vdash \longrightarrow A \vdash$}
\RightLabel{$[\neg R]$}
\UnaryInfC{$\vdash \neg\Box\neg A \longrightarrow A \vdash$}
\RightLabel{$[sym]$}
\UnaryInfC{$\vdash \neg\Box\neg A \longleftarrow A \vdash$}
\RightLabel{$[\Box R]$}
\UnaryInfC{$A \vdash \Box\neg\Box\neg A$}
\end{prooftree}
$$

Another example delabelling

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{x : \neg A, x : A \vdash}\,[\neg L]}{Ryx, y : \Box\neg A, x : A \vdash}\,[\Box L]}{Ryx, x : A \vdash y : \neg\Box\neg A}\,[\neg R]}{Rxy, x : A \vdash y : \neg\Box\neg A}\,[sym]}{x : A \vdash x : \Box\neg\Box\neg A}\,[\Box R]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\Box\neg A \vdash \longrightarrow A \vdash}\,[\Box L]}{\vdash \neg\Box\neg A \longrightarrow A \vdash}\,[\neg R]}{\vdash \neg\Box\neg A \longleftarrow A \vdash}\,[sym]}{A \vdash \Box\neg\Box\neg A}\,[\Box R]
$$

Another example delabelling

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{\color{red}{x : \neg A, x : A \vdash}}\,[\neg L]}{Ryx, y : \Box\neg A, x : A \vdash}\,[\Box L]}{Ryx, x : A \vdash y : \neg\Box\neg A}\,[\neg R]}{Rxy, x : A \vdash y : \neg\Box\neg A}\,[sym]}{x : A \vdash x : \Box\neg\Box\neg A}\,[\Box R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\color{red}{\neg A, A \vdash}}\,[\neg L]}{\Box\neg A \vdash \longrightarrow A \vdash}\,[\Box L]}{\vdash \neg\Box\neg A \longrightarrow A \vdash}\,[\neg R]}{\vdash \neg\Box\neg A \longleftarrow A \vdash}\,[sym]}{A \vdash \Box\neg\Box\neg A}\,[\Box R]$$

Another example delabelling

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{x : \neg A, x : A \vdash}\,[\neg L]}{\color{red}{Ryx, y : \Box\neg A, x : A \vdash}}\,[\Box L]}{Ryx, x : A \vdash y : \neg\Box\neg A}\,[\neg R]}{Rxy, x : A \vdash y : \neg\Box\neg A}\,[sym]}{x : A \vdash x : \Box\neg\Box\neg A}\,[\Box R]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\color{red}{\Box\neg A \vdash \longrightarrow A \vdash}}\,[\Box L]}{\vdash \neg\Box\neg A \longrightarrow A \vdash}\,[\neg R]}{\vdash \neg\Box\neg A \longleftarrow A \vdash}\,[sym]}{A \vdash \Box\neg\Box\neg A}\,[\Box R]
$$

Another example delabelling

$$
\begin{array}{cl}
x : A \vdash x : A & \\ \hline
x : \neg A, x : A \vdash & [\neg L] \\ \hline
Ryx, y : \Box\neg A, x : A \vdash & [\Box L] \\ \hline
\color{red}{Ryx, x : A \vdash y : \neg\Box\neg A} & [\neg R] \\ \hline
Rxy, x : A \vdash y : \neg\Box\neg A & [sym] \\ \hline
x : A \vdash x : \Box\neg\Box\neg A & [\Box R]
\end{array}
$$

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\neg A, A \vdash & [\neg L] \\ \hline
\Box\neg A \vdash \longrightarrow A \vdash & [\Box L] \\ \hline
\color{red}{\vdash \neg\Box\neg A \longrightarrow A \vdash} & [\neg R] \\ \hline
\vdash \neg\Box\neg A \longleftarrow A \vdash & [sym] \\ \hline
A \vdash \Box\neg\Box\neg A & [\Box R]
\end{array}
$$

Another example delabelling

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{x : \neg A, x : A \vdash}\,[\neg L]}{Ryx, y : \Box\neg A, x : A \vdash}\,[\Box L]}{Ryx, x : A \vdash y : \neg\Box\neg A}\,[\neg R]}{\color{red}{Rxy, x : A \vdash y : \neg\Box\neg A}}\,[sym]}{x : A \vdash x : \Box\neg\Box\neg A}\,[\Box R]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\Box\neg A \vdash \longrightarrow A \vdash}\,[\Box L]}{\vdash \neg\Box\neg A \longrightarrow A \vdash}\,[\neg R]}{\color{red}{\vdash \neg\Box\neg A \longleftarrow A \vdash}}\,[sym]}{A \vdash \Box\neg\Box\neg A}\,[\Box R]
$$

Another example delabelling

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A}{x : \neg A, x : A \vdash}\,[\neg L]}{Ryx, y : \Box\neg A, x : A \vdash}\,[\Box L]}{Ryx, x : A \vdash y : \neg\Box\neg A}\,[\neg R]}{Rxy, x : A \vdash y : \neg\Box\neg A}\,[sym]}{\color{red}{x : A \vdash x : \Box\neg\Box\neg A}}\,[\Box R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\Box\neg A \vdash \longrightarrow A \vdash}\,[\Box L]}{\vdash \neg\Box\neg A \longrightarrow A \vdash}\,[\neg R]}{\vdash \neg\Box\neg A \longleftarrow A \vdash}\,[sym]}{\color{red}{A \vdash \Box\neg\Box\neg A}}\,[\Box R]$$

Tree Hypersequent Rules: Modal Operators

$$\dfrac{\mathcal{H}[X \vdash Y \curvearrowright X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \curvearrowright X' \vdash Y']}\ [\Box L] \qquad \dfrac{\mathcal{H}[X \vdash Y \curvearrowright\ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\dfrac{\mathcal{H}[X \vdash Y \curvearrowright A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \dfrac{\mathcal{H}[X \vdash Y \curvearrowright X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \curvearrowright X' \vdash Y']}\ [\Diamond R]$$

Forms of Cut

$$\frac{\mathcal{H}[X \vdash A, Y] \qquad \mathcal{H}[X, A \vdash Y]}{\mathcal{H}[X \vdash Y]}\ [Cut^{a}]$$

Forms of Cut

$$\frac{\mathcal{H}[X \vdash A, Y] \qquad \mathcal{H}[X, A \vdash Y]}{\mathcal{H}[X \vdash Y]} \, [Cut^{a}]$$

$$\frac{\mathcal{H}[X \vdash A, Y] \qquad \mathcal{H}'[X, A \vdash Y]}{(\mathcal{H} \oplus \mathcal{H}')[X \vdash Y]} \, [Cut^{m}]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad\qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X' \vdash Y' \longrightarrow X \vdash Y]}\ [eKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \longrightarrow X' \vdash Y']}\ [eKR]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X' \vdash Y' \frown X \vdash Y]}\ [eKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \frown X' \vdash Y']}\ [eKR]$$

$$\mathcal{H}[X, A \vdash A, Y] \qquad [axK]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\,[iWR]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iWL] \qquad\qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\ [iWR]$$

$$\frac{\mathcal{H}[X'' \vdash Y'' \longleftarrow X \vdash Y \longrightarrow X' \vdash Y']}{\mathcal{H}[X' \vdash Y' \longrightarrow X', X'' \vdash X', Y'']}\ [eWo]$$

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\,[iWR]$$

$$\frac{\mathcal{H}[X'' \vdash Y'' \curvearrowleft X \vdash Y \curvearrowright X' \vdash Y']}{\mathcal{H}[X' \vdash Y' \curvearrowright X', X'' \vdash X', Y'']}\,[eWo]$$

$$\frac{\mathcal{H}[X'' \vdash Y'' \curvearrowright X \vdash Y \curvearrowleft X' \vdash Y']}{\mathcal{H}[X \vdash Y \curvearrowleft X', X'' \vdash X', Y'']}\,[eWi]$$

A cut elimination theorem for tree hypersequent systems is relatively straightforward.

One option is a contraction-free style argument (by Negri and von Plato), following the construction for Labelled Sequent systems.

Another is the Curry–Belnap argument.

Virtues and Vices

	DISPLAY	LABELLED	DELABELLED
Cut-free	+	+	+
Explicit	+	+	+
Systematic	+	+	+
Separation	+	+	+
Subformula	+	+−	+
Nonredundant	−	+−	+
Gentzen-plus	−	+−	+

Virtues and Vices

	DISPLAY	LABELLED	DELABELLED
Cut-free	$+$	$+$	$+$
Explicit	$+$	$+$	$+$
Systematic	$+$	$+$	$+$
Separation	$+$	$+$	$+$
Subformula	$+$	$+-$	$+$
Nonredundant	$-$	$+-$	$+$
Gentzen-plus	$-$	$+-$	$+$

Display Logic, Labelled Sequents and Hypersequents

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THANK YOU!

http://blogs.unimelb.edu.au/logic/

@standefer on Twitter

Based on NASSLLI 2016 slides by Greg Restall and Shawn Standefer.

https://consequently.org/class/2016/PTPLA-NASSLLI/