An Introduction to Proof Theory

Class 1: Foundations

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Our Aim

To introduce *proof theory*, with a focus on its applications in philosophy, linguistics and computer science.

Introduce the basics of sequent systems and Gentzen's *Cut Elimination Theorem*.

Sequents

Left and Right Rules

Structural Rules

Cut Elimination

Consequences

Onward to Classical Logic

Another approach to Cut Elimination

Proofs and Models

SEQUENTS

Gerhard Gentzen

Natural deduction to sequents

$$\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad A^{[1]}}{B \to C}\,[\to E] \qquad B}{C}\,[\to E]}{A \to C}\,[\to I]\ 1$$

- $A \to (B \to C), A \vdash B \to C$
- $A \to (B \to C), A, B \vdash C$
- $A \to (B \to C), B \vdash A \to C$

Sequents record consequences of premises

Lay out relations explicitly

Natural deduction to sequents

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$$\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad A^{[1]}}{B \to C}\,[\to E] \qquad B}{C}\,[\to E]}{A \to C}\,[\to I]\ 1$$

- $A \to (B \to C), A \vdash B \to C$
- $A \to (B \to C), A, B \vdash C$
- $A \to (B \to C), B \vdash A \to C$

Sequents record consequences of premises

Lay out relations explicitly

Sequents

$$X \vdash A$$

X is a *sequence*
Could also use *sets*, *multisets*, or more *general structures*

Sequent proofs

Rather than introduction and elimination rules,
sequent systems use *left* and *right* introduction rules

Proofs are trees built up by rules.

There are two sorts of rules: *Connective rules* and *structural rules*

LEFT AND RIGHT RULES

Left and right rules

$$\frac{X, A, Y \vdash C}{X, A \land B, Y \vdash C}\ [\land L_1] \qquad \frac{X, A, Y \vdash C \quad U, B, V \vdash C}{X, U, A \lor B, Y, V \vdash C}\ [\lor L]$$

$$\frac{X, B, Y \vdash C}{X, A \land B, Y \vdash C}\ [\land L_2] \qquad \frac{X \vdash A}{X \vdash A \lor B}\ [\lor R_1]$$

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \land B}\ [\land R] \qquad \frac{X \vdash B}{X \vdash A \lor B}\ [\lor R_2]$$

Left and right rules

$$\frac{X \vdash A}{X, \neg A \vdash}\,[\neg L] \qquad \frac{X \vdash A \qquad Y, B, Z \vdash C}{Y, X, A \to B, Z \vdash C}\,[\to L]$$

$$\frac{X, A \vdash}{X \vdash \neg A}\,[\neg R] \qquad \frac{X, A \vdash B}{X \vdash A \to B}\,[\to R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{p \wedge r \vdash p}\,[\wedge L_1]}{p \wedge r \vdash p \vee q}\,[\vee R_1] \qquad \dfrac{q \vdash q}{q \vdash p \vee q}\,[\vee R_2]}{(p \wedge r) \vee q \vdash p \vee q}\,[\vee L] \qquad s \vdash s}{(p \wedge r) \vee q, s \vdash (p \vee q) \wedge s}\,[\wedge R]$$

$$\dfrac{\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\,[\neg L]}{p \vdash \neg\neg p}\,[\neg R]}{\vdash p \rightarrow \neg\neg p}\,[\rightarrow R]$$

STRUCTURAL RULES

Identity axiom

$$p \vdash p$$

What about arbitrary formulas in the axioms?

Either prove a theorem or take generalizations as axioms

$$A \vdash A$$

Weakening

$$\frac{X, Y \vdash C}{X, A, Y \vdash C}\,[KL] \qquad \frac{X \vdash}{X \vdash A}\,[KR]$$

Contraction

$$\frac{X, A, A, Z \vdash C}{X, A, Z \vdash C}\,[WL]$$

Permutation

$$\frac{X, A, B, Z \vdash C}{X, B, A, Z \vdash C}\ [CL]$$

Cut

$$\frac{X \vdash A \qquad Y, A, Z \vdash B}{Y, X, Z \vdash B}\ [\text{Cut}]$$

The system with all the connective rules, the axiom rule, and the structural rules [KL], [KR], [CL], [WL] will be LJ

LJ+Cut will be LJ with the addition of [Cut]

Sequent Proof

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p \vdash p}{q, p \vdash p}\ [KL]}{p \wedge q, p \vdash p}\ [\wedge L_2]}{p, p \wedge q \vdash p}\ [CL]}{p \wedge q, p \wedge q \vdash p}\ [\wedge L_1]}{p \wedge q \vdash p}\ [WL]
$$

$$
\cfrac{\cfrac{p \vdash p}{p, \neg p \vdash}\ [\neg L]}{p, \neg p \vdash q}\ [KR]
$$

Cut is the only rule in which formulas *disappear* going from premiss to conclusion

A proof is *Cut-free* iff it does not contain an application of the Cut rule

If you know there is a Cut-free derivation of a sequent, it can make finding a proof easier

CUT ELIMINATION

Hauptsatz

Gentzen called his Elimination Theorem the *Hauptsatz*

He showed that for sequent derivable with a Cut, there is a Cut-free derivation

Admissibility and derivability

$$\frac{S_1, \ldots, S_n}{S}\ [R]$$

A rule [R] is *derivable* iff given derivations of $S_1, \ldots, S_n$,
one can extend those derivations to obtain a derivation of S

A rule [R] is *admissible* iff if there are derivations of $S_1, \ldots, S_n$,
then there is a derivation of S

Admissibility and derivability

Consider the system LJ.

The rule

$$\frac{X, A, B \vdash C}{X, A \wedge B \vdash C}\ [\wedge L_3]$$

is *derivable* in LJ

The Elimination Theorem shows that Cut is *admissible* in LJ, even though it is not derivable

Theorem

If there is a derivation of $X \vdash A$ in $\mathrm{LJ} + \mathrm{Cut}$,
then there is a Cut-free derivation of $X \vdash A$

Auxiliary concepts

In the Cut rule,

$$\frac{(L)\quad X \vdash A \qquad Y, A, Z \vdash B \quad (R)}{(C)\quad Y, X, Z \vdash B}\,[\mathrm{Cut}]$$

the displayed A is the *cut formula*

There are two ways of measuring the complexity of a Cut: *grade* and *rank*

The *grade*, γ, of an application of [Cut] is the number of logical symbols in the cut formula A.

The *left rank*, ρ_L, of an application of [Cut] is the length of the longest path starting with (L) containing A in the succeedent

The *right rank*, ρ_R, of an application of [Cut] is the length of the longest path starting with (R) containing A in the antecedent

The *rank*, ρ, of an application of [Cut] is $\rho_L + \rho_R$

Proof setup

Double induction on grade and rank of a Cut

Outer induction is on grade, inner induction is on rank

Proof strategy

Show how to move Cuts above rules,
lowering left rank, then right rank,
then lowering grade

Parametric Cuts are cuts in which
the Cut formula is not the one displayed in a rule

Principal Cuts are ones in which
the Cut formula is the one displayed in a rule

If one premiss of a Cut comes via an axiom or a weakening step,
then the Cut can be eliminated entirely

Eliminating Cuts: Parametric

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{X' \vdash A}}{X \vdash A}[\sharp] \qquad \dfrac{\vdots\,\pi_2}{A, Y \vdash C}}{X, Y \vdash C}[\mathrm{Cut}]$$

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{A, Y' \vdash C}}{A, Y \vdash C}[\flat]}{X, Y \vdash C}[\mathrm{Cut}]$$

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{X' \vdash A} \qquad \dfrac{\vdots\,\pi_2}{A, Y \vdash C}}{X', Y \vdash C}[\mathrm{Cut}]}{X, Y \vdash C}[\sharp]$$

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\vdots\,\pi_2}{A, Y' \vdash C}}{X, Y' \vdash C}[\mathrm{Cut}]}{X, Y \vdash C}[\flat]$$

Eliminating Cuts: Parametric

$$
\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{X, A \vdash C} \qquad \dfrac{\vdots\,\pi_2}{Y, B \vdash C}}{X, Y, A \vee B \vdash C}\,[\vee L] \qquad \dfrac{\vdots\,\pi_3}{C, Z \vdash D}}{X, Y, A \vee B, Z \vdash D}\,[\mathrm{Cut}]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{X, A \vdash C} \qquad \dfrac{\vdots\,\pi_3}{C, Z \vdash D}}{X, A, Z \vdash D}\,[\mathrm{Cut}] \qquad \dfrac{\dfrac{\vdots\,\pi_2}{Y, B \vdash C} \qquad \dfrac{\vdots\,\pi_3}{C, Z \vdash D}}{Y, B, Z \vdash D}\,[\mathrm{Cut}]}{X, Y, A \vee B, Z, Z \vdash D}\,[\vee L]}{X, Y, A \vee B, Z \vdash D}\,[WL]
$$

Eliminating Cuts: Principal

$$
\AxiomC{$\vdots\ \pi_1$}
\noLine
\UnaryInfC{$X \vdash A$}
\RightLabel{[$\vee$R]}
\UnaryInfC{$X \vdash A \vee B$}
\AxiomC{$\vdots\ \pi_2$}
\noLine
\UnaryInfC{$A, Y \vdash C$}
\AxiomC{$\vdots\ \pi_3$}
\noLine
\UnaryInfC{$B, Z \vdash C$}
\RightLabel{[$\vee$L]}
\BinaryInfC{$A \vee B, Y, Z \vdash C$}
\RightLabel{[Cut]}
\BinaryInfC{$X, Y, Z \vdash C$}
\DisplayProof
$$

$$
\AxiomC{$\vdots\ \pi_1$}
\noLine
\UnaryInfC{$X \vdash A$}
\AxiomC{$\vdots\ \pi_2$}
\noLine
\UnaryInfC{$A, Y \vdash C$}
\RightLabel{[Cut]}
\BinaryInfC{$X, Y \vdash C$}
\RightLabel{[KL]}
\UnaryInfC{$X, Y, Z \vdash C$}
\DisplayProof
$$

Eliminating Cuts: Principal

$$\dfrac{\dfrac{\begin{matrix}\vdots\, \pi_1\\ X, A \vdash B\end{matrix}}{X \vdash A \to B}\,\text{[}\to R\text{]} \qquad \dfrac{\begin{matrix}\vdots\, \pi_2 \\ U \vdash A\end{matrix} \qquad \begin{matrix}\vdots\, \pi_3\\ Y, B, Z \vdash C\end{matrix}}{Y, U, A \to B, Z \vdash C}\,\text{[}\to L\text{]}}{Y, U, X, Z \vdash C}\,\text{[Cut]}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, \pi_2\\ U \vdash A\end{matrix} \qquad \begin{matrix}\vdots\, \pi_1\\ X, A \vdash B\end{matrix}}{X, U \vdash B}\,\text{[Cut]} \qquad \begin{matrix}\vdots\, \pi_3\\ Y, B, Z \vdash C\end{matrix}}{Y, X, U, Z \vdash C}\,\text{[Cut]}}{Y, U, X, Z \vdash C}\,\text{[CL]}$$

Eliminating Cuts: Special Cases

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash p\end{array} \qquad p \vdash p}{X \vdash p}\,[\mathrm{Cut}]
\qquad\qquad
\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ Y \vdash C\end{array}}{A, Y \vdash C}\,[\mathrm{KL}]}{X, Y \vdash C}\,[\mathrm{Cut}]$$

$$\begin{array}{c}\vdots\,\pi_1\\ X \vdash p\end{array}
\qquad\qquad
\dfrac{\begin{array}{c}\vdots\,\pi_2\\ Y \vdash C\end{array}}{X, Y \vdash C}\,[\mathrm{KL}]$$

Contraction

Contraction causes some problems for this proof

$$\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\overset{\vdots\,\pi_2}{A, A, Y \vdash C}}{A, Y \vdash C}\,[WL]}{X, Y \vdash C}\,[\mathrm{Cut}]$$

$$\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \overset{\vdots\,\pi_2}{A, A, Y \vdash C}}{X, A, Y \vdash C}\,[\mathrm{Cut}]}{X, X, Y \vdash C}\,[\mathrm{Cut}]$$

Solution

Use a stronger rule that removes *all* copies of the formula in one go

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y^{-A} \vdash B}\,[\mathrm{Mix}]$$

Y is required to contain at least one copy of A

We can extend the proof to cover contraction by proving that Mix is admissible

The admissibility of Mix has the admissibility of Cut as a corollary

Mix cases

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{A, A, Y \vdash C}}{A, Y \vdash C}\,[WL]}{X, Y^{-A} \vdash C}\,[Mix]$$

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\vdots\,\pi_2}{A, A, Y \vdash C}}{X, Y^{-A} \vdash C}\,[Mix]$$

Eliminating Mix: Complications with rank

$$
\begin{prooftree}
\AxiomC{$\vdots\, \pi_1$}
\noLine
\UnaryInfC{$X, A \vdash$}
\RightLabel{[$\neg$R]}
\UnaryInfC{$X \vdash \neg A$}
\AxiomC{$\vdots\, \pi_2$}
\noLine
\UnaryInfC{$\neg A, Y \vdash A$}
\RightLabel{[$\neg$L]}
\UnaryInfC{$\neg A, Y, \neg A \vdash$}
\RightLabel{[Mix]}
\BinaryInfC{$X, Y^{-\neg A} \vdash$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots\, \pi_1$}
\noLine
\UnaryInfC{$X, A \vdash$}
\RightLabel{[$\neg$R]}
\UnaryInfC{$X \vdash \neg A$}
\AxiomC{$\vdots\, \pi_1$}
\noLine
\UnaryInfC{$X, A \vdash$}
\RightLabel{[$\neg$R]}
\UnaryInfC{$X \vdash \neg A$}
\AxiomC{$\vdots\, \pi_2$}
\noLine
\UnaryInfC{$\neg A, Y \vdash A$}
\RightLabel{[Mix]}
\BinaryInfC{$X, Y^{-\neg A} \vdash A$}
\RightLabel{[$\neg$L]}
\UnaryInfC{$X, Y^{-\neg A}, \neg A \vdash$}
\RightLabel{[Mix]}
\BinaryInfC{$X, X^{-\neg A}, Y^{-\neg A} \vdash$}
\RightLabel{[WL]}
\UnaryInfC{$X, Y^{-\neg A} \vdash$}
\end{prooftree}
$$

Eliminating Mix: Complications with grade

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ X, A \vdash\end{array}}{X \vdash \neg A}\,[\neg R] \qquad \dfrac{\begin{array}{c}\vdots\ \pi_2\\ Y \vdash A\end{array}}{Y, \neg A \vdash}\,[\neg L]}{X, Y \vdash}\,[\mathrm{Mix}]$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_2\\ Y \vdash A\end{array} \qquad \begin{array}{c}\vdots\ \pi_1\\ X, A \vdash\end{array}}{Y, X^{-A} \vdash}\,[\mathrm{Mix}]}{Y, X \vdash}\,[\mathrm{KL}]}{X, Y \vdash}\,[\mathrm{CL}]$$

CONSEQUENCES

Subformula property

In rules besides Cut, all formulas
appearing in the premises appear in the conclusion

This is the *Subformula Property*

In Cut-free derivations, formulas not appearing in the end sequent don't appear in the rest of the proof, which makes proof search easier

Conservative extension

One sequent system T^+ is a *conservative extension* of another sequent system T, just in case the language of T^+ extends that of T, and if $X \vdash A$ is derivable in T^+ then $X \vdash A$ is derivable in T, when X, A are in the language of T.

The Elimination Theorem yields conservative extension results via the Subformula Property.

If X and A are all in the base language, then the Subformula Property guarantees that a proof of $X \vdash A$ in T^+ will not use any of the rules not available in T.

Consistency

In the presence of [KL] and [KR], $\emptyset \vdash \emptyset$ says everything implies everything.

The Elimination Theorem implies that that is not provable

Suppose that it is. There is then a Cut-free derivation.
All the axioms have formulas on both sides,
and no rules delete formulas.
So there is no derivation of $\emptyset \vdash \emptyset$.

Unprovability results

Similar arguments can be used to show that $\vdash p \vee \neg p$ isn't derivable.

How would a Cut-free derivation go?
The last rule would have to be $[\vee R]$,
applied to either $\vdash p$ or $\vdash \neg p$,
neither of which is provable

Disjunction property

Suppose that $\vdash A \vee B$ is derivable

There is a Cut-free derivation,
so the last rule has to be $[\vee R]$.
So either $\vdash A$ or $\vdash B$ is derivable.

ONWARD TO CLASSICAL LOGIC

A seemingly magical fact

LJ is complete for *intuitionistic logic*

A sequent system for classical logic, LK, can be obtained by allowing the succedent to contain more than one formula

$A_1, \ldots, A_k \vdash B_1, \ldots, B_n$ says that if all the A_is hold, then one of the B_js does too.

Ian Hacking remarked that this seemed magical, and it was explored in Peter Milne's paper "Harmony, Purity, Simplicity, and a 'Seemingly Magical Fact'"

Left and right rules

$$\frac{X, A, Y \vdash Z}{X, A \wedge B, Y \vdash Z}\ [\wedge L_1]$$

$$\frac{X, B, Y \vdash Z}{X, A \wedge B, Y \vdash Z}\ [\wedge L_2]$$

$$\frac{X \vdash Y, A, Z \qquad U \vdash V, B, W}{X, U \vdash Y, V, A \wedge B, Z, W}\ [\wedge R]$$

$$\frac{X, A, Y \vdash Z \qquad U, B, V \vdash W}{X, U, A \vee B, Y, V \vdash Z, W}\ [\vee L]$$

$$\frac{X \vdash Y, A, Z}{X \vdash Y, A \vee B, Z}\ [\vee R]$$

$$\frac{X \vdash Y, B, Z}{X \vdash Y, A \vee B, Z}\ [\vee R]$$

Left and right rules

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ [\neg L] \qquad \frac{X \vdash Y, A, Z \qquad U, B, V \vdash W}{U, X, A \to B, V \vdash Y, Z, W}\ [\to L]$$

$$\frac{X, A \vdash Y}{X \vdash \neg A, Y}\ [\neg R] \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\ [\to R]$$

Weakening

$$\frac{X \vdash Y}{A, X \vdash Y}\,[KL]$$

$$\frac{X \vdash Y}{X \vdash Y, A}\,[KR]$$

Contraction

$$\frac{X, A, A, Z \vdash Y}{X, A, Z \vdash Y}\ [WL]$$

$$\frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z}\ [WR]$$

Permutation

$$\frac{X, A, B, Z \vdash Y}{X, B, A, Z \vdash Y}\ [CL]$$

$$\frac{X \vdash Y, A, B, Z}{X \vdash Y, B, A, Z}\ [CR]$$

Classical proofs

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\vdash p, \neg p}\,[\neg R]}{\vdash p \vee \neg p, \neg p}\,[\vee R_1]}{\vdash p \vee \neg p, p \vee \neg p}\,[\vee R_2]}{\vdash p \vee \neg p}\,[WR]
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{q \vdash q}{q, \neg q \vdash}\,[\neg L]}{q \wedge \neg q, \neg q \vdash}\,[\wedge L_1]}{q \wedge \neg q, q \wedge \neg q \vdash}\,[\wedge L_2]}{q \wedge \neg q \vdash}\,[WL]
$$

Some features

An Elimination Theorem is provable for LK

Since LK can have multiple formulas on the right,
one can apply [WR] as well as the connective rules
as the final rule in a proof of $\vdash A$

Consequently, LK does not have the Disjunction Property

ANOTHER APPROACH TO CUT ELIMINATION

Alternatives

Different ways of setting up a sequent system may lead
to different ways to prove the Elimination Theorem

One way, explored by Dyckhoff, Negri and von Plato, originally due to Dragalin, is to *absorb* the structural rules into the connective rules

There are no structural rules in this system,
but their effects are implicit in the connective rules

Instead of sequences in the sequents, we will use multisets

Rules

Identity axiom: $X, p \vdash p, Y$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ [\wedge L]$$

$$\frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ [\wedge R]$$

$$\frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ [\vee R]$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ [\vee L]$$

Weakening Admissibility: If $X \vdash Y$ is provable in n steps, then $X' \vdash Y'$ is provable in at most n steps, where $X \subseteq X'$, $Y \subseteq Y'$

Inversion Lemma: If the conclusion of a rule is provable in n steps, then the premiss of the rule is provable in at most n steps

Contraction Admissibility: If $A, A, X \vdash Y$ is provable in n steps, then $A, X \vdash Y$ is; and if $X \vdash Y, A, A$ is provable in at most n steps, then $X \vdash Y, A$ is.

These are *height-preserving admissibility* lemmas

Elimination Theorem

One can show Cut is admissible

Since there are no contraction rules, we do not have to use Mix

Since there are fewer rules, there are fewer cases to check

PROOFS AND MODELS

Invalid sequents and positions

Call an unprovable sequent $X \nvdash Y$ a *position*.

Write as $[X : Y]$.

Can use failed proof search to generate models.

How does one get from *proof* to *truth*?

Models are ways of systematically elaborating finite positions into ideal, infinite positions that settle every proposition

In the propositional case, valuations are generated by ideal positions

Positions to models

$$[X : Y]$$

The members of X are *true* and the members of Y are *false*

Positions to models

$$[X : Y]$$

The members of X are *true* and the members of Y are *false* (relative to $[X : Y]$).

Example

$$[p \vee q, r : \neg p]$$

Example

$$[p \lor q, r : \neg p]$$

$$\frac{}{p \lor q, r}$$

$$\frac{}{\quad}$$

Example

$$[p \vee q, r : \neg p]$$

$$\dfrac{}{p \vee q, r \quad \textit{true}}$$

$$\overline{}$$

Example

$$[p \vee q, r : \neg p]$$

$$\begin{array}{ll} \hline p \vee q, r & \textit{true} \\ \neg p & \\ & \\ \hline \end{array}$$

Example

$$[p \lor q, r : \neg p]$$

$p \lor q, r$	*true*
$\neg p$	*false*

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	???

Example

$$[p \lor q, r : \neg p]$$

$p \lor q, r$	*true*
$\neg p$	*false*
p	???

DEFINITION: A is *true* at $[X : Y]$ iff $X \vdash A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \vdash Y$.

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	*true*

DEFINITION: A is *true* at $[X : Y]$ iff $X \vdash A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \vdash Y$.

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	*true*
$p \wedge r$	

DEFINITION: A is *true* at $[X : Y]$ iff $X \vdash A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \vdash Y$.

Example

$$[p \vee q, r : \neg p]$$

$p \vee q, r$	*true*
$\neg p$	*false*
p	*true*
$p \wedge r$	*true*

DEFINITION: A is *true* at $[X : Y]$ iff $X \vdash A, Y$.

DEFINITION: A is *false* at $[X : Y]$ iff $X, A \vdash Y$.

Classical Logic

$A \wedge B$ is true at $[X : Y]$ iff A and B are true at $[X : Y]$.

$A \vee B$ is false at $[X : Y]$ iff A and B are false at $[X : Y]$.

$\neg A$ is true at $[X : Y]$ iff A is false at $[X : Y]$.

$\neg A$ is false at $[X : Y]$ iff A is true at $[X : Y]$.

$$A \wedge B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].$$

$$A \vee B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].$$

$$\neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y].$$

$$\neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y].$$

However, $p \wedge q$ is false at $[\ : p \wedge q]$

$A \wedge B$ is true at $[X : Y]$ iff A and B are true at $[X : Y]$.

$A \vee B$ is false at $[X : Y]$ iff A and B are false at $[X : Y]$.

$\neg A$ is true at $[X : Y]$ iff A is false at $[X : Y]$.

$\neg A$ is false at $[X : Y]$ iff A is true at $[X : Y]$.

However, $p \wedge q$ is false at $[\ : p \wedge q]$ but neither p nor q is false at $[\ : p \wedge q]$ since neither $p \vdash p \wedge q$ nor $q \vdash p \wedge q$.

$$A \wedge B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].$$

$$A \vee B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].$$

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Similarly, r is neither true nor false at $[p : q]$.

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$$\frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y}\,[\text{Cut}]$$

Maximal Sequents

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FACT: If $X \not\models Y$, there's a *maximal* $[\mathcal{X} : \mathcal{Y}]$ extending $[X : Y]$.

Assign truth values relative to *maximal* positions.

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In a slogan, *truth value = location in a maximal position*,

Variations

The ideal position construction handles classical logic

With some small adjustments, it can be used to provide models for intuitionistic logic

The system LJ is single-conclusion, but there is a intuitionistic sequent system that has multiple conclusions

The construction with these two systems yield *Kripke models* and *Beth models*

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Substructural Logics and their Proof Theory

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Based on NASSLLI 2016 slides by Greg Restall and Shawn Standefer.

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