Noncommutative resolutions and Coulomb branches

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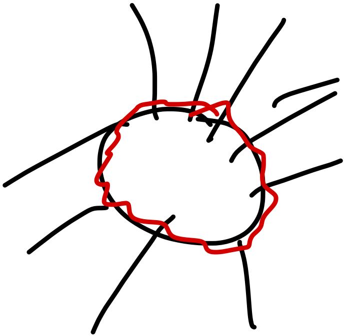
Let $X = T^*\mathrm{Fl}_n$ be the cotangent bundle of the flag variety $X_0 = \mathrm{Fl}_n$ over a field $\Bbbk$ of characteristic $p \geq 0$.

Let $\mathsf{Coh}_0(X)$ denote the abelian category of coherent sheaves on X which are (set-theoretically) supported on X_0.



Consider the algebra $A = U\mathfrak{gl}_n(\mathbb{k})$. Let $A\text{-mod}_0$ be the principal block of the category of finite dimensional modules with central character.

HC

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Theorem (Bezrukavnikov-Mirković)

If $p \gg 0$, there is an equivalence of derived categories

$$D^b(\mathsf{Coh}_0(X)) \cong D^b(A\text{-mod}_0).$$

Bezrukavnikov calls this a “non-commutative counterpart of the Springer resolution.”

This is a beautiful equivalence, but it’s quite abstract. I want to give you a somewhat more concrete way of thinking about it.

$\mathsf{Coh}_0(X)$	$A\text{-mod}_0$
geometry	representation theory

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$\mathsf{Coh}_0(X)$

geometry

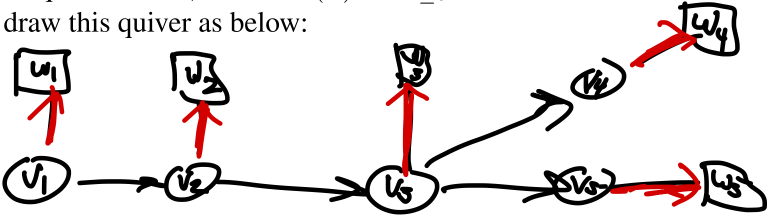
A-mod_0

representation theory

$\mathring{R}$-mod_0

combinatorial algebra

Let Γ be a quiver, and $\mathbf{v}, \mathbf{w} \colon I = \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ a dimension vector. We usually draw this quiver as below:



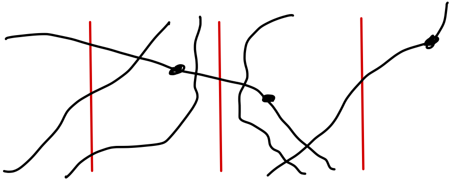
The example relevant to $T^* \operatorname{Fl}_n$ is:

Note that $T^* \operatorname{Fl}_n$ is the Nakajima quiver variety for this dimension vector. There's a different way of constructing $T^* \operatorname{Fl}_n$ from this quiver though: Coulomb branches. To describe these, I need to introduce KLRW algebras.

Definition

A (planar) KLRW diagram is a generic collection of curves in $\mathbb{R} \times [0,1]$ of the form $\{(\pi(t), t) \mid t \in [0,1]\}$ for $\pi\colon [0,1] \to \mathbb{R}$.

1. Each strand is labeled from [redacted] Γ and is black or red. There are v_i black strands and w_i red strands with label i.
2. Red strands must be vertical at fixed, distinct x-values (for example, $x = 1/W, 2/W, \dots, 1$ for $W = \sum w_i$).
3. We place dots at a finite number of points on black strands, avoiding crossings.



Definition

A cylindrical KLRW diagram is a generic collection of curves in $\mathbb{R}/\mathbb{Z} \times [0, 1]$ of the form $\{(\pi(t), t) \mid t \in [0, 1]\}$ for $\pi\colon [0, 1] \to \mathbb{R}/\mathbb{Z}$.

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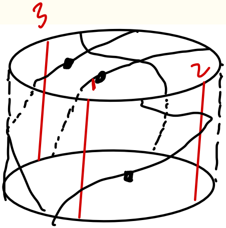
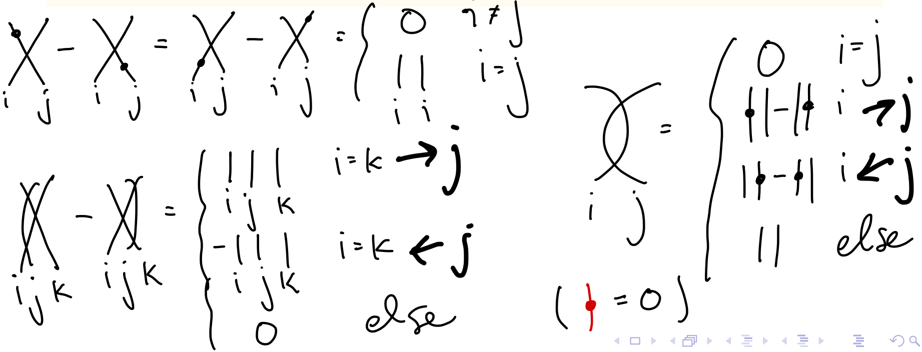
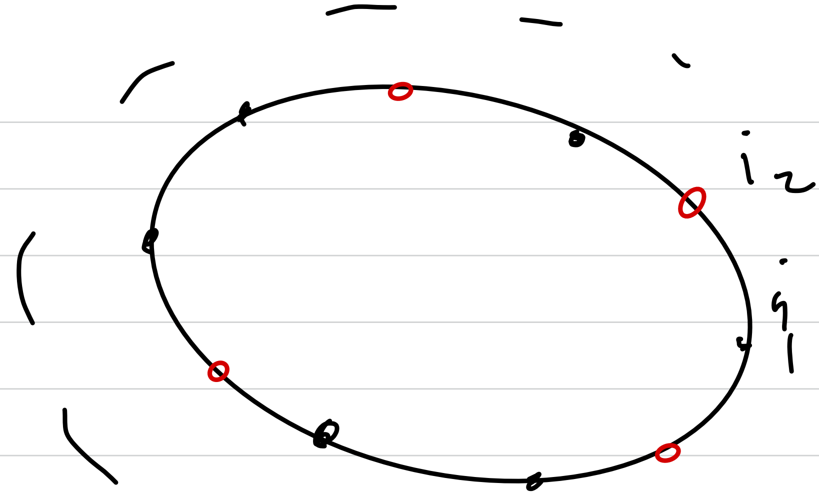


We can compose KLRW diagrams by stacking, if the labels on the bottom of one and top of the other match up to isotopy (never moving red strands).

Definition

The (planar) KLRW algebra R is the formal $\Bbbk$-span of planar KLRW diagrams modulo the local relations below.




i2
i1


L1
L2

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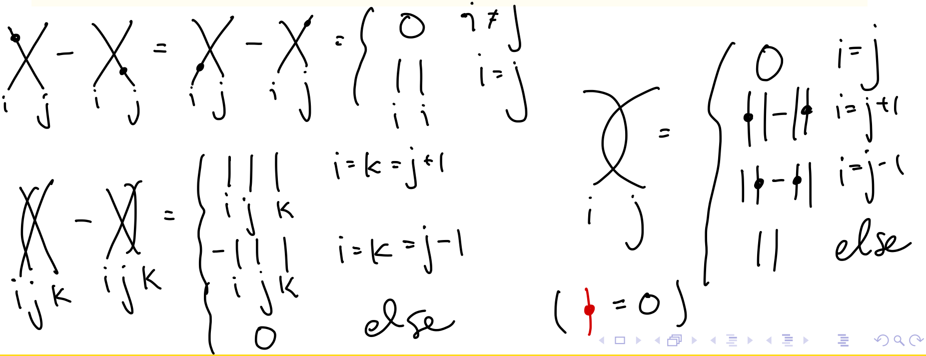
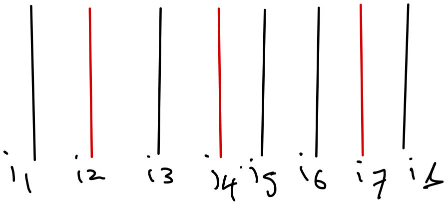


KLRW algebras

Important role is played by idempotents where all strands are vertical.



There's one of these for each possible order on strands. Can encode this in a word $\mathbf{i}$ in $\Gamma \cup \Gamma$. Denote by $e(\mathbf{i})$.

Definition

The (planar) KLRW category is the category whose objects are words as above, and where $\mathrm{Hom}(\mathbf{i}, \mathbf{j}) = e(\mathbf{j})\tilde{R}e(\mathbf{i})$.

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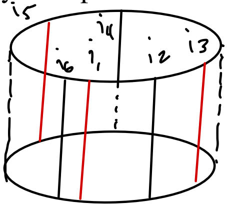

There's one of these for each possible order on strands. Can encode this in a word $\mathbf{i}$ in $\Gamma \cup \Gamma$. Denote by $e(\mathbf{i})$.

For $\mathring{R}$, this word is really cyclic, but can always start with red at $x = 0$.

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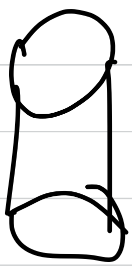
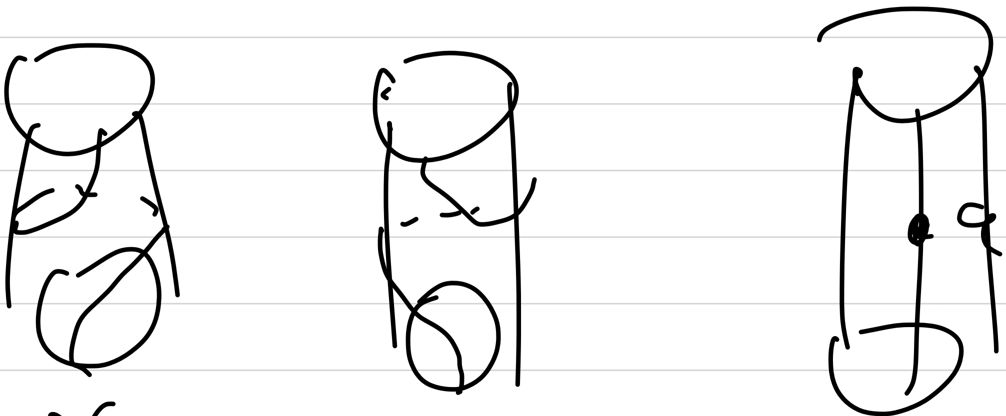


$$\mathbb{C}[T^*\mathbb{C}^*] \cong \mathbb{C}[x, x^{-1}, a]$$



$x \qquad x^{-1}$

$$H_*^{\mathbb{C}^*}(Gr_{\mathbb{C}^*})$$
$$\wr$$
$$\mathbb{Z}$$

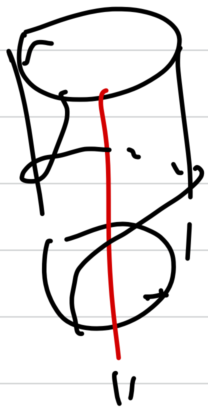
$a \in H^*_{\mathbb{C}^*}(pt)$

$x = [t]$

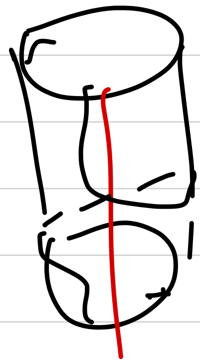
$x^{-1} = [t^{-1}]$

$\mathbb{C}[r_1, r_{-1}]$

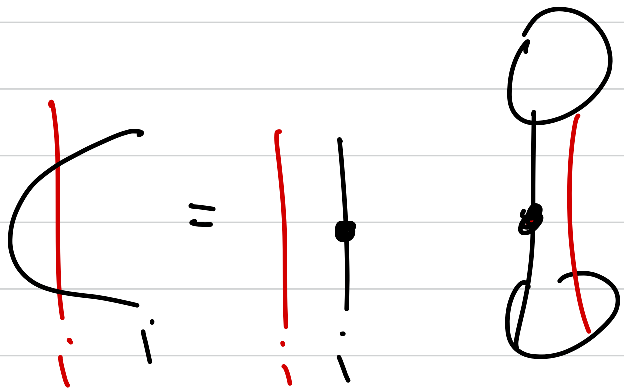
$r_1 r_{-1} = a$



$a, \quad r_1$



r_{-1}



Planar KLR algebra ($w_i = 0$) has a natural geometric interpretation. Let

$$N_{\mathbf{v}} = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \qquad G_{\mathbf{v}} = \prod_i GL(\mathbb{C}^{v_i})$$

The quotient $Y_{\mathbf{v}} = N_{\mathbf{v}}/G_{\mathbf{v}}$ is the moduli space of quiver representations of dimension $\mathbf{v}$.

Consider a word $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$ where $i \in I$ appears v_i times. We say that a homogeneous complete flag F_k on $\bigoplus_{i \in I} \mathbb{C}^{v_i}$ has type $\mathbf{i}$ if

$$\dim(F_k \cap \mathbb{C}^{v_j}) = \dim(F_{k-1} \cap \mathbb{C}^{v_j}) + \delta_{j,i_k}.$$

Let $X_{\mathbf{i}}$ be the moduli space of quiver representations equipped with a flag of subrepresentations of type $\mathbf{i}$. $X_{\mathbf{i}} = \tilde{N}_{\mathbf{i}}/B$

Basic geometric object to consider:

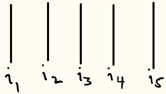
$$R_{\mathbf{v}} = \bigoplus_{\mathbf{i},\mathbf{j}} H_*^{BM}(X_{\mathbf{i}} \times_{Y_{\mathbf{v}}} X_{\mathbf{j}}) \cong \mathrm{Ext}^*\Big(\bigoplus_{\mathbf{i}} \pi_* \mathbb{C}_{X_{\mathbf{i}}}\Big)$$

as an algebra under convolution.

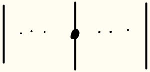
Theorem (Varagnolo-Vasserot, Rouquier)

The algebra $R_{\mathbf{v}}$ is generated by the homology classes: ≅ KLR

- *the diagonal in $X_{\mathbf{i}} \times_{Y_{\mathbf{v}}} X_{\mathbf{i}}$*



- *the 1st Chern class of the tautological bundle*



F_k/F_{k-1}

- *push-pull from a partial flag version*

modulo the relations from before.

To generalize KLR algebra, need to give more geometric definition:

Consider a generic cocharacter $\xi\colon \mathbb{C}^* \to G$; we have a resulting complete flag $\{F_w\}$ of some type $\mathbf{i}$ given by the sum of vectors of weight $\leq w$ for each w.

Let $N_{\mathbf{i}}^-$ be the elements of $N_{\mathbf{v}}$ of negative weight under ξ. Let $P_{\mathbf{i}}^- \subset G_{\mathbf{v}}$ be the subgroup preserving the flag $F_\bullet$.

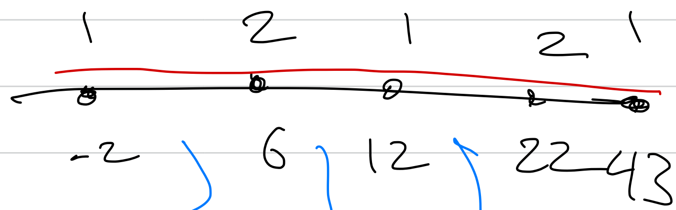
Proposition

We have an isomorphism $X_{\mathbf{i}} \cong N_{\mathbf{i}}^-/P_{\mathbf{i}}^-$.

Pushforward $X_\xi \to Y$ generalizes "spiral induction" of Lusztig and Yun.

$$\begin{bmatrix} t^{a_{1,1}} \; -2 & & & \\ & t^{a_{1,2}} \; 12 & & \\ & & \ddots & \\ & & & t^{a_{1,v_1}} \; 43 \end{bmatrix}$$

$$\begin{bmatrix} t^{a_{2,1}} = 6 & & \\ & \ddots & \\ & & 22 \\ & & t^{a_{2,v_2}} \end{bmatrix}$$



Can do this for **any** representation N and group G.

Important twist: can consider the different $\xi\colon \mathbb{C}^* \to \mathrm{Norm}_{GL(V)}(G)$ which lift a fixed $\mathbb{C}^*$-action on $Y = N/G$.

Can similarly define $X_\xi = N_\xi^- / P_\xi^-$, and consider

$$R = \bigoplus_{\xi,\xi'} H_*^{BM}(X_\xi \times_Y X_{\xi'}) \cong \mathrm{Ext}^*(\bigoplus_{\xi} \pi_* \mathbb{C}_{X_\xi}).$$

Of course, there are infinitely many ξ, but only finitely many N_ξ^- up to conjugacy.

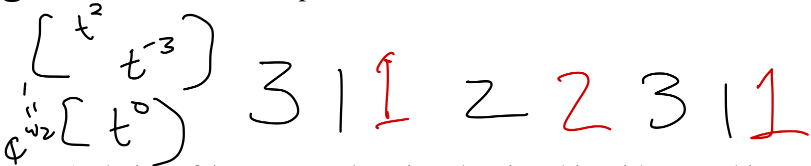
Theorem (Sauter, W.)

The algebra R always has a "KLR-type" presentation.

To obtain the algebras of ultimate interest to us, we have to add w_i copies of the representation $\mathbb{C}^{v_i}$ to $N_{\mathbf{v}}$ (i.e. $\mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i})$). This is sometimes called "framing."

Moduli spaces of framed quiver representations are closely related to Nakajima quiver varieties.

We'll want to choose $\varphi\colon \mathbb{C}^* \to \prod GL(\mathbb{C}^{w_i}) \subset \mathrm{Aut}_G(N_{\mathbf{v}}^{\mathbf{w}})$. This puts an order on the basis vectors of the $\mathbb{C}^{w_i}$'s, which we can record as a word in I with w_i copies of i.



A choice of ξ corresponds to interleaving this with a word in I containing v_i copies of i.

Theorem (W.)

For framed quiver representations, the Ext algebra R is the planar KLRW algebra discussed above.

I was interested in these algebras to construct categorifications of tensor products and knot invariants, and their connections to quiver varieties.

I'll say more about this later, but let me just mention that the bimodules that correspond to braiding can be gotten by taking Ext between pushforwards for different φ's.

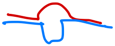
But then I learned that there was a quite different lens to view them through: Coulomb branches.

Now affinize everything:

Taylor series $\mathsf{C} = \mathbb{C}[[t]]$	$\mathsf{G} = G[[t]]$	$\mathsf{N} = N[[t]]$
Laurent series $\mathcal{C} = \mathbb{C}((t))$	$\mathcal{G} = G((t))$	$\mathcal{N} = N((t))$

Relevant spaces:

$$\mathsf{Y} = \mathsf{N}/\mathsf{G} = \mathrm{Map}(D = \operatorname{Spec} \mathsf{C} \to N/G)$$



$$\mathcal{Y} = \mathcal{N}/\mathcal{G} = \mathrm{Map}(D^* = \operatorname{Spec} \mathcal{C} \to N/G)$$

These can be interpreted as spaces of principal G bundles with a section of the associated N-bundle on D and D^*.

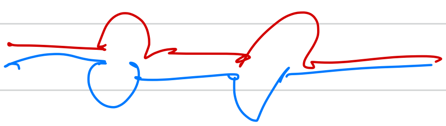
Thus, the fiber product $\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}$ is the space of such bundles on the "raviolo" gluing two copies of D along D^*.

Choose trivialization on f-disk

$$\frac{N[[t]] \times G((t))}{G[[t]]} \cong \frac{N_{\mathcal{O}} \times G_{K}}{G_{\mathcal{O}}}$$

$$\downarrow$$

$$G_{\mathcal{O}} \circlearrowright N_{\mathcal{O}} \longrightarrow N_{K}$$



Previous experience tells us it would be fun to consider

$$A = H_*^{BM}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}).$$

Using factorization arguments, we can see that A is a commutative $\mathbb{C}$-algebra of finite type.

Definition

The **Coulomb branch** *is the spectrum* $\mathfrak{M} = \operatorname{Spec} A$.

This definition has some motivation in 3d QFT (it's the local operators in a topological twist of a gauge theory), but it's also recognizable as an affine version of our construction of KLR algebras, In the case of $N = N_{\mathbf{v}}$ and $G = G_{\mathbf{v}}$ from before,

- $\mathcal{N}/\mathcal{G}$ is the moduli of quiver representations over the field $\mathcal{C}$.
- N/G is the moduli of such quiver representations with a choice of lattice $\Lambda_i \cong \mathsf{C}^{v_i} \subset \mathcal{C}^{v_i}$ that gives a subrepresentation.

But our KLR presentation comes from being able to switch consecutive spaces in a flag, so we want flags, not lattices.

Definition

An **affine flag** *in $\mathcal{C}^m$ is a sequence of a lattices $\mathsf{F}_k \subset \mathcal{C}^m$ for $k \in \mathbb{Z}$ such that*

$$\cdots \subset \mathsf{F}_{k-1} \subset \mathsf{F}_k \subset \mathsf{F}_{k+1} \subset \cdots \qquad t\mathsf{F}_k = \mathsf{F}_{k-m}$$

Objects describing affine flags are periodic (periodic permutations for Schubert cells, etc.)

$$\mathbb{C}((t))^m = \langle e_1, \ldots, e_m \rangle$$

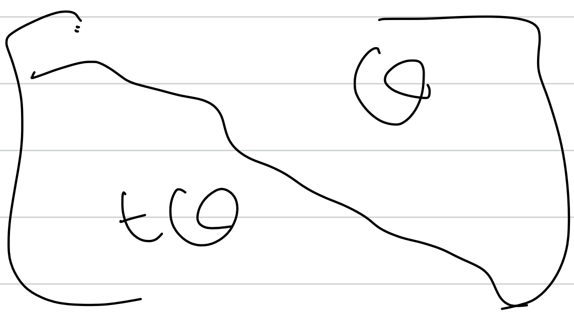
$$\cup$$

$$\langle te_m, e_1, \ldots, e_{m-1} \rangle$$

$$\langle te_{m-1}, te_m, \ldots, e_{m-2} \rangle$$

$$\vdots$$

$$\langle te_1, \ldots, te_m \rangle$$



So, we can now let i be a **periodic word**: a map $\mathsf{i}\colon \mathbb{Z} \to I$ such that $\mathsf{i}_k = \mathsf{i}_{k+m}$ for all k for $m = \sum v_i$ such that any m consecutive entries contain v_i copies of i.

Any homogeneous affine flag $\mathsf{F}_\bullet \subset \bigoplus_{i \in I} \mathcal{C}^{v_i}$ has a periodic word as its type, defined by

$$\dim(\mathsf{F}_k \cap \mathcal{C}^{v_j} / \mathsf{F}_{k-1} \cap \mathcal{C}^{v_j}) = \delta_{j,\mathsf{i}_k}.$$

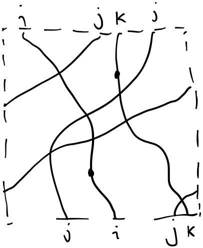
Let X_i be the moduli space of quiver reps over $\mathcal{C}$, together with a choice of affine flag of subreps of type i.

Theorem

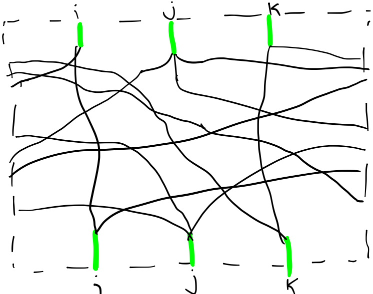
The convolution algebra

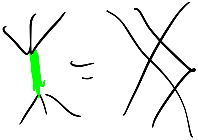
$$\mathring{R} = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM}(\mathsf{X}_\mathsf{i} \times_{\mathcal{Y}} \mathsf{X}_\mathsf{j}) \cong \operatorname{Ext}^*\Big(\bigoplus_{\mathsf{i}} \pi_* \mathbb{C}_{X_\mathsf{i}}\Big)$$

is the cylindrical KLRW algebra.



To get the Coulomb branch A, need to integrate out the finite flag variety back down to a single lattice. This corresponds to having a thick strand bringing together all with label i for each i at top and bottom of diagram (but general cKLRW diagram in the middle).





This result generalizes to the KLRW case with addition of red strands.

reps over $\mathbb{C}$ w/ flags → planar KLRW

quiver Γ → reps over $\mathbb{C}((t))$ w/ affine flags → cylindrical KRLW

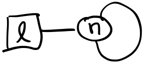
reps over $\mathbb{C}((t))$ w/ lattices → Coulomb branches

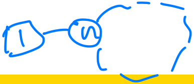
Why am I interested in this construction? Mainly because lots of examples recover interesting varieties, always symplectic:

- my favorite: nilcone of $\mathfrak{gl}_n$ ← T^*Fl_n

- more generally, Slodowy slices in type A

- symmetric power $\mathrm{Sym}^n(\mathbb{C}^2/\mathbb{Z}_\ell)$





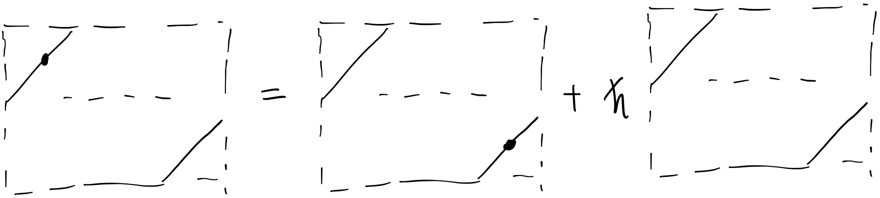
$$\begin{bmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{-2\pi i/\ell} \end{bmatrix}$$

All of these varieties are Nakajima quiver varieties, but for potentially different quivers and dimension vectors. The quiver variety and Coulomb branch for a given quiver should be related by “3d mirror symmetry”/“symplectic duality.”

The disk D has a $\mathbb{C}^*$ action by rotation (so the parameter t has weight 1). Combining this with the action on N/G via φ, we obtain compatible $\mathbb{C}^*$-actions on $\mathsf{Y}, \mathcal{Y}, \mathsf{X}_{\mathsf{i}}$.

$$A_\hbar = H_*^{BM,\mathbb{C}^*}(\mathsf{Y} \times_{\mathcal{Y}} \mathsf{Y}). \qquad \mathring{R}_\hbar = \bigoplus_{\mathsf{i},\mathsf{j}} H_*^{BM,\mathbb{C}^*}(\mathsf{X}_{\mathsf{i}} \times_{\mathcal{Y}} \mathsf{X}_{\mathsf{j}})$$

Relations only change to account for the fact that $\mathsf{F}_k/\mathsf{F}_{k-1}$ and $\mathsf{F}_{k+m}/\mathsf{F}_{k+m-1}$ are isomorphic, but have different $\mathbb{C}^*$-weight.



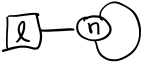
The parameters $H_*^{BM,\mathbb{C}^*}(\mathsf{Y}) \cong \mathbb{C}[\mathfrak{g}, \hbar]^G$ give a maximal commutative subalgebra S of $A_\hbar$.

These result in well-known quantizations of these varieties; we get more familiar algebras if we consider the specialization A_1 setting $\hbar = 1$.

- my favorite: $U(\mathfrak{sl}_n)$ with S the Gelfand-Tsetlin subalgebra

- more generally, W-algebras in type A

- spherical Cherednik algebras for S_n or $G(\ell, 1, n)$, with S the subalgebra generated by the Dunkl-Opdam operators.



Definition

We call an A_1-module M **Gelfand-Tsetlin** *if the subalgebra S acts locally finitely on M, i.e.* $\dim(S \cdot m) < \infty$ *for all $m \in M$.*

Theorem

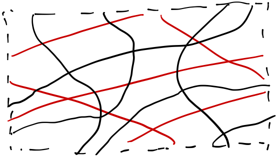
The category of Gelfand-Tsetlin A_1-modules with "integral weights" is equivalent to the category of weakly graded (gradeable after passing to associated graded) R-modules.

So, passing to GT A_1-modules undoes the affinization!

A few words on the proof:

- a GT module satisfies $M = \bigoplus_{\gamma \in \mathrm{MaxSpec}(S)} W_\gamma(M)$ for
$$W_\gamma(M) = \{m \in M \mid \mathfrak{m}_\gamma^N m = 0 \text{ for all } N \gg 0\}.$$
Note that we can think of $\gamma \in \mathrm{MaxSpec}(S)$ as a conjugacy class of cocharacters $\mathbb{C}^* \to N_{GL(V)}(G)$. (integrality!)
- The category is thus controlled by natural transformations $W_\gamma \to W_{\gamma'}$.
- We have an isomorphism (by localization in equivariant homology)
$$\mathrm{Hom}(W_\gamma, W_{\gamma'}) \cong H_*^{BM}(X_{\gamma'} \times_Y X_\gamma)$$
- This gives the desired R-action.

For any φ, φ', there is a bimodule relating the two different quantizations where we wrap the red lines around the cylinder the appropriate number of times.



Derived tensor product with this bimodule gives "twisting functors."

This is a special case of a construction for all symplectic resolutions.

Theorem

Twisting functors give a (finite) braid group action on the categories of modules over different quantizations.

This can be upgraded to an action of tangles; the resulting link homology $\mathscr{D}_q(K)$ recovers my old work on categorified Reshetikhin-Turaev (in particular, Khovanov-Rozansky in type A).

In type A, can even upgrade this to an action of the foam category.

This seems to be a version of Witten's prediction of a knot homology constructed with A-branes on a space of Hecke modifications.

Applications:

- Gives Koszul duality between categories $\mathcal{O}$ for Coulomb branches and quiver varieties/hyperkähler quotients attached to a given (G,N).
- First classification of GT modules for $\mathfrak{gl}_n$, and character formulae for them (Kamnitzer-Tingley-W.-Weekes-Yacobi, Silverthorne-W.).
- analogous classification for modules over Cherednik algebras of $G(\ell,p,n)$. (LePage-W.)
- Categorified knot invariants have two Koszul dual constructions; Coulomb side construction seems to be "A-branes on Hecke modifications" proposed by Witten.

I'm supposed to be telling you about non-commutative resolutions.

BFN construct (usual) resolutions of singularities: Springer resolution and Hilbert scheme. This works for (G, N) an affine type A quiver gauge theory, but not in most other cases.

Definition

A **noncommutative symplectic resolution** *of a symplectic singularity* $\mathfrak{M} = \operatorname{Spec} A$ *is a ring* R *such that* $A = eRe$ *for some idempotent, and the functor* $M \mapsto eM \colon R\text{-mod} \to A\text{-mod}$ *"looks like" pushforward by a crepant resolution of singularities.*

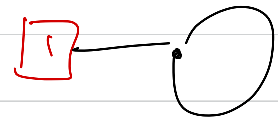
For a symplectic singularity, symplectic resolution=crepant resolution.

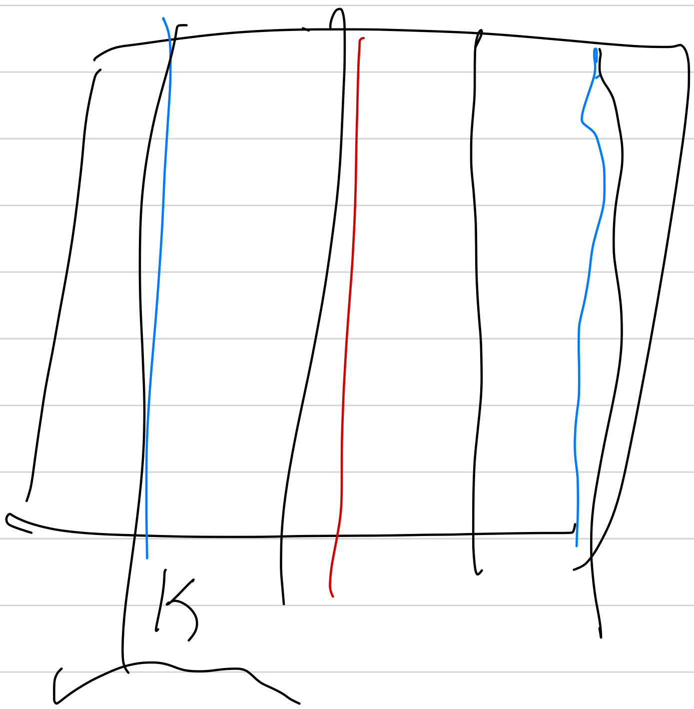
Theorem

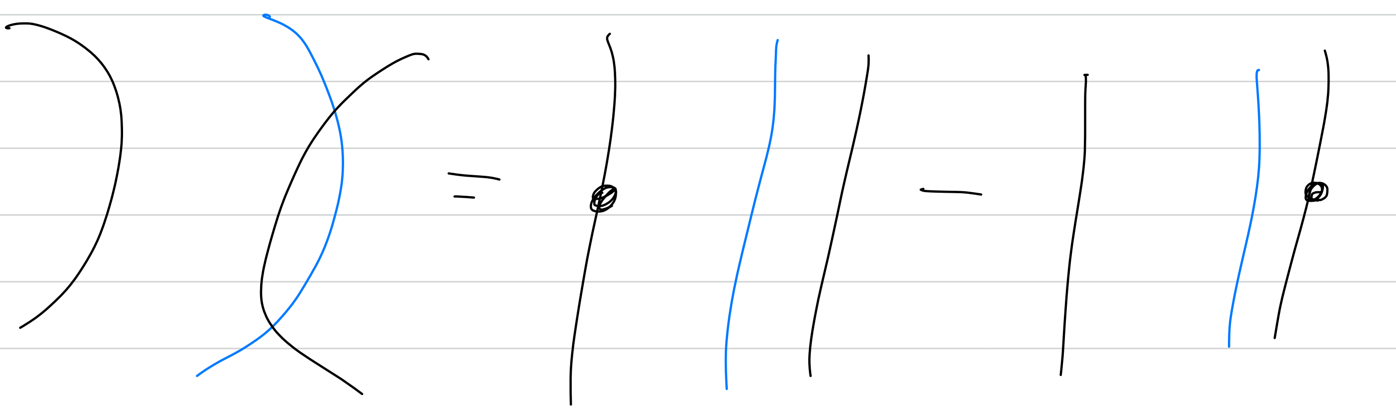
Whenever a BFN resolution exists, the ring $\mathring{R}$ is a non-commutative symplectic resolution of A and $D^b(\mathring{R}\text{-mod}) \cong D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}))$ for $\mathfrak{M}$ any symplectic resolution of the Coulomb branch $\operatorname{Spec} A$.

- "Noncommutative Springer resolution" in type A is a special case; this gives such resolutions for all parabolic Slodowy slices in type A.
- In the case of Hilbert scheme (or more generally, affine type A) need to account for extra $\mathbb{C}^*$ acting by scaling on the loop (symplectic $\mathbb{C}^*$ on $\mathbb{C}^2$). Need to use "weighted" version of $\mathring{R}$. Recovers BFG resolution based on Cherednik algebra.

$k \in \mathbb{F}_p \overset{1/p}{\hookrightarrow} \mathbb{R}/\mathbb{Z}$ $\qquad GL_n \circlearrowright Mat_{n\times n} \oplus \mathbb{C}^n \circlearrowleft \mathbb{C}^*$







For different people, this next part will have different motivations:

- You might want to understand coherent sheaves on a resolution of $\operatorname{Spec} A$.
- You might be the kind of person who says "what if $\mathbb{k}$ had characteristic p"?
- You might have gone to some recent talks of Aganagić and gotten confused once cigars came up.

Interestingly, either way, you should do the same thing.

Over $\mathbb{F}_p$, you can try to analyze finite dimensional modules over A_1 by diagonalizing S again. Again, let's restrict to integral maximal ideals.

Problem?

If we wrap a strand around the cylinder p times, the shift of the dot is trivial.

Theorem

Let $\Bbbk = \mathbb{F}_p$. For generic φ (and p big enough), the category of finite dimensional A_1-modules with "integral weights" is equivalent to the category of finite dimensional weakly graded $\mathring{R}$-modules.

So, still affinized, but resolved now.

Similar arguments to last time:

- geometric proof: localization to μ_p-fixed points on Y.
- algebraic proof: same calculations as last time, but now we have natural transformations $W_{\mathsf{i},0} \to W_{\mathsf{j},0}$ as endomorphisms given by any diagram on the cylinder with i where all strands have winding number divisible by p.

Why does this have anything to do with coherent sheaves?

Fancy char p stuff: there's a quantum Frobenius map $A_0 \to Z(A_1)$. This is actually the sections of a map of sheaves $\mathscr{O}_{\tilde{\mathfrak{M}}} \to \mathscr{Q}_{\tilde{\mathfrak{M}}}$ of structure sheaf to a localization of A_1 on any resolution $\tilde{\mathfrak{M}}$.

Applying results of Bezrukavnikov and Kaledin, we can construct a very special vector bundle $\mathcal{T}$ on $\tilde{\mathfrak{M}}$ by "diagonalizing the action of $S \subset A_1$."

A lift of this vector bundle also exists in char 0, so can forget about characteristic p story.

A **tilting generator** is a vector bundle T such that $\mathrm{Ext}^{>0}(T, T) = 0$, and $\langle T \rangle = D^b\mathsf{Coh}(\tilde{\mathfrak{M}})$.

Theorem

Assume that G is a torus, or (G, N) corresponds to an affine type A quiver gauge theory. The vector bundle $\mathcal{T}$ is a tilting generator for $\tilde{\mathfrak{M}}$ and $\mathrm{End}(\mathcal{T}) = \mathring{R}$.

The fact that $\mathring{R}$ is a non-commutative resolution is a corollary.

While all commutative and non-commutative resolutions are derived equivalent, these equivalences are not unique. Instead, they generate an action of the affine braid group on this category; these descend from twisting functors in char p:

Theorem

The twisting affine braid group action on $D^b(\mathring{R}\text{-mod})$ *is generated by cylindrical versions of R-matrix bimodules.*

In fact, this extends to an action of affine tangles, by affine versions of the cup and cap bimodules, and in type A to affine foams. This gives a link homology $\mathscr{D}_{coh}(K)$.

Theorem

The following link homologies are all the same:

- $\mathscr{D}_{coh}(K)$, *constructed from the affine tangle action above.*
- $\mathscr{D}_q(K)$, *constructed from the tangle action on quantum coherent sheaves.*
- *the invariant constructed in my older knot homology work (which matches Khovanov-Rozansky in type A).*
- *Aganagić's physical construction.*

Thanks for listening.