# Exercise for Chapter 10: Functions on $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$

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1. Let $\varphi \in P_k$, where $P_k$ is the space of homogeneous polynomials of degree k on $\mathbb{R}^n$. Given the formula:

$$\Delta(r^2\varphi) = 2(n+2k)\varphi + r^2\Delta\varphi$$

By induction on $s$, see which of the following formula is true for $\varphi \in P_{k-2s}$:

$$\Delta(r^{2s}\varphi) = 2s(n+2k-2s-2)r^{2s-2}\varphi + r^2 s\Delta\varphi$$

or

$$\Delta(r^{2s}\varphi) = 2s(n+2k-4s-4)r^{2s-2}\varphi + r^2 s\Delta\varphi$$

2. Consider three operators on the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$:

$$e = \frac{1}{2}\Delta, \quad h = r\frac{\partial}{\partial r} + \frac{n}{2}, \quad f = \frac{1}{2}r^2$$

verify they satisfy the following commutation relation:

$$[h, e] = -2e, \quad [h, f] = 2f, \quad [e, f] = h.$$

Convince yourself that $r^k$ is the lowest-weight vector of weight $k + \frac{n}{2}$ in the spaces $E_k$. Try your best to build some understanding of what the lowest-weight spaces of $\mathbb{C}[x_1, \cdots, x_n]$ look like.