1. Recap Polar decomposition
2. Recap Gram-Schmidt process
3. Bruhat decomposition

Polar decomposition

- Polar decomposition decomposes $GL_n\mathbb{R}$ into positive definite matrices times the orthogonal matrices.

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- Then $e^A = se^D s^{-1}$, where $e^D = diag(e^{\lambda_1}, ..., e^{\lambda_n})$.
- So e^A has positive eigenvalues and so is positive definite.

Remark on polar decomposition

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The polar decomposition $GL_n(\mathbb{R}) \cong P \times O_n$ can be generalised to ant lie group G with finitely many connected components. The subgroup O_n is gneralised to a maximal compact subgroup K of G, and $P = \exp \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement with respect to the Killing form of the Lie algebra K in that of G. Will see more in Chapter 8.

The Gram-Schmidt process

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Define $T = U_n \cap B$ where B is the group of complex upper triangular matrices. Then T is diagonal by orthogonality.

Theorem

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The map is $uT \to uB$, and $gB = ubB = uB \to uT$.
$U_n/T \cong GL_n\mathbb{C}/B$ is isomorphic to the space of flags in $\mathbb{C}^n$.
Recall a space of flags is a sequence of subspaces
$E_1 \subset E_2 \subset ... \subset E_n = \mathbb{C}^n$ with $\dim(E_k) = k$.
They are very important in representation of U_n and $GL_n\mathbb{C}$.

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2. Recap Gram-Schmidt process
3. Bruhat decomposition

Bruhat decomposition

- The Bruhat decomposition decomposes $G = \cup_{w \in W} BwB$, where G is a reductive group, B is a Borel subgroup, and W is the Weyl group.
- Here we will see $GL_n\mathbb{C}/B = \cup_{\pi \in S_n} N\pi B$, where B is the upper triangular matrices, N is a subgroup of the upper triangular matrices, S_n is the symmetric group.

Gaussian eliminations

- Let $g \in GL_n(\mathbb{C})$, conduct two column operations represented by right multiplication of

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & -b & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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- $c_i \to c_i - bc_j$, with $j < i$, i.e. subtract from a given column a multiple of any column to its left.
- So we get a matrix w in reduced echelon form: where each column w_i ends in 1 in $\pi_1 th$ column, and
- the entries to the right of each 1 vanishes: $w_{\pi,j} = 0$ if $j > i$.

Permutation

- Each column w_i ends in 1 in $\pi_i th$ column.
- For $w = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
- we have $\pi_1 = 3, \pi_2 = 1, \pi_3 = 4, \pi_4 = 2$.

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$g = wb = n\pi b$ where b is an upper triangular matrix, π is a permutation matrix, n is an upper triangular matrix with $1's$ on the diagonal.

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Any element $g \in GL_n(\mathbb{C})$, can be factorised into $g = n\pi b$, where n belongs to the subgroup N of upper triangular matrices with 1's on the diagonal, π is a permutation matrix, and b belongs to the subgroup B of upper-triangular matrices.

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eg. $g = \begin{pmatrix} 6 & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{c_2 \to c_2 - 2c_1} \begin{pmatrix} 6 & -9 \\ 1 & 0 \end{pmatrix} \xrightarrow{c_2 \to -\frac{1}{9}c_2} \begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{c_1 \leftrightarrow c_2}$ $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$. $gb'\pi^{-1} = \begin{pmatrix} 6 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{9} \\ 0 & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} = n$

$g = \begin{pmatrix} 6 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -9 \end{pmatrix} = n\pi b$. Note that the decomposition $g = n\pi b$ where $n \in N$ is not unique.

Uniqueness

- w can be characterised as: $n = w\pi^{-1} \in N$ (column operation before); and $\pi^{-1}w \in \tilde{N}$, where $\tilde{N}$ are the lower triangular matrices. This is the row operation on w which gives the transpose of the other.
- If $l := \pi^{-1}w$, then $w = n\pi = \pi l$. So $n = \pi l \pi^{-1}$.
- By setting $n \in N \cap \pi\tilde{N}\pi^{-1}$, the unique factorization is unique.

Theorem

The decomposition $g = n\pi b$ is unique if n is chosen in $N \cap \pi\tilde{N}\pi^{-1}$.

Cor4.6

- Elements in $GLn\mathbb{C}/B$ are $gB = nbB = nB$. So the orbit of the N acting on $GLn\mathbb{C}/B$ is labelled by π . So there are $|S_n| = n!$ orbits.

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- Fix $gB \in GLn\mathbb{C}/B$, the orbit of gB under N is $NgB = \{ngB|n \in N\} = \{n_1\pi bB|n_1 \in N\} = \{n_2\pi B|n \in N \cap \pi\tilde{N}\pi^{-1}\}$, to ensure the uniqueness (not counting repeatedly). As g and so π is fixed, $NgB \cong N_\pi$.

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- It seems obvious to others that the number of variables entries in w is l_π, and so $N_\pi \cong \mathbb{C}^{l_\pi}$.

Corollary (4.6)

The orbit of N on the flag manifold $GL_n\mathbb{C}/B$ decompose it into $n!$ cells C_π, with $C_\pi \cong N_\pi \cong \mathbb{C}^{l_\pi}$.

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- By Lagrange theorem, $|G/B| = |G|/|B| = \prod_{k=1}^{q} \frac{q^k-1}{q-1}$.
- $|N_\pi| = q^{l_\pi}$.
- So the Bruhat decomposition gives:

$$|G/B| = \sum_{\pi \in S_n} |N_\pi| \tag{3.1}$$

$$\prod_{k=1}^{q} \frac{q^k - 1}{q - 1} = \sum_{\pi \in S_n} q_\pi^l \tag{3.2}$$