IRREDUCIBILITY CRITERION FOR STANDARD MODULES

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These notes are written for the Representation Theory Seminar at the University of Melbourne in April, 2022. They are mostly based on [Hec+].

Be careful: in these notes I tried to translate notations and conventions from [Hec+] to [DV], which is a daunting task. There are potentially a lot of mistakes, especially on "positive" versus "negative" and on various ρ-shifts. Please refer to the original paper if you want to apply the statements.

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Set:

- G - a connected semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$.
- θ - an involution of $\mathfrak{g}$ (the Cartan involution in the classical setting).
- K - a reductive algebraic subgroup of G such that its Lie algebra $\mathfrak{k}$ is the fixed point of θ.

In the classical setting, one starts with a real connected linear semisimple Lie group G_0 with a choice of maximal compact subgroup K_0 and a Cartan involution Θ. Then G and K will be the complexifications of G_0 and K_0, and θ will be the differential of Θ.

Remark 0.1. To study non-linear groups, one instead require K to be an algebraic group over $\mathbb{C}$ with an isogeny to $\operatorname{Int}\mathfrak{g}$.

Geometric objects and weights will have the following conventions:

- $X = \mathcal{B}$ - the flag variety of $\mathfrak{g}$. For $x \in X$, $\mathfrak{b}_x$ denotes the corresponding Borel subalgebra in $\mathfrak{g}$.
- $\mathfrak{h}$ - abstract Cartan algebra of $\mathfrak{g}$. $\Phi \supseteq \Phi^-$ denote the root system and negative roots corresponding to roots in the Borel.
- $\mathcal{D}_{X,\lambda}$ - G-homogeneous twisted sheaf of differential operators corresponding to $\lambda \in \mathfrak{h}^*$ parameterized such that $\mathcal{D}_X = \mathcal{D}_{X,0}$ and $\mathcal{D}_{\omega_X} = \mathcal{D}_{X,-2\rho}$.
- If $Q \subset X$ is a K-orbit with inclusion map j_Q, and if τ is a K-equivariant connection on Q, the standard module on Q is $\mathcal{I}(Q,\tau) = j_{Q*}\tau$, and the irreducible on Q is $\mathcal{L}(Q,\tau) = j_{Q!*}\tau$.

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The main goal of these talks is to present an irreducibility criterion for standard modules $\mathcal{I}(Q,\tau)$. It can be shown that $\Gamma(X,\mathcal{I}(Q,\tau))^\vee$ are equal to the standard representations in Knapp-Zuckerman's version of Langlands classification [KZ77, Theorem 5], and the reducibility of those modules for regular infinitesimal characters were classified by Speh-Vogan [SV80]. Speh-Vogan covered the singular case to certain degree (only a necessary condition was explicitly written down) but it is much more complicated compared to the regular case. Milicic suspected that this could be a $\mathcal{D}$-module result, and went ahead to prove it together with other people in [Hec+, 8.8].

The statement of the result is a bit involved. I plan to first discuss some special cases. The general case will be a merge of the special cases. The argument is mostly taken from *op. cit.*

1. Preliminaries

First I want to recall some basic facts on different types of roots and how they are related to K-orbits on X.

1.1. **Different types of roots.** Let $\mathfrak{c} \subseteq \mathfrak{g}$ be a Cartan subalgebra stable under the Cartan involution θ. Then $\mathfrak{c}$ decomposes as $\mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{c}^\theta$ (the toroidal/compact part) and $\mathfrak{a} = \mathfrak{c}^{-\theta}$ (the split part). Also θ acts on the root system $\Phi_\mathfrak{c} = \Phi(\mathfrak{g},\mathfrak{c})$. A root α is called

- **imaginary** if $\theta\alpha = \alpha$ (equivalently $\alpha|_\mathfrak{a} = 0$);
- **real** if $\theta\alpha = -\alpha$ (equivalently $\alpha|_\mathfrak{t} = 0$);
- **complex** in all other cases.

We can subdivide the imaginary roots into two subcases: an imaginary root α is called

- **compact imaginary** if the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}$, i.e. $\theta \circlearrowright \mathfrak{g}_\alpha$ by 1;
- **noncompact imaginary** if the root space $\mathfrak{g}_\alpha$ is not contained in $\mathfrak{k}$. Then necessarily $\theta \circlearrowright \mathfrak{g}_\alpha$ by -1.

Let

$$\Phi_{\mathfrak{c},CI}, \quad \Phi_{\mathfrak{c},NI}, \quad \Phi_{\mathfrak{c},\mathbb{R}}, \quad \Phi_{\mathfrak{c},\mathbb{C}} \tag{1.1}$$

denote the sets of compact imaginary, noncompact imaginary, real and complex roots, respectively.

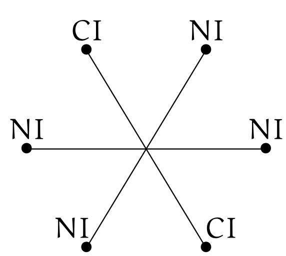
Although all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\operatorname{Ad} G$, in order to fix θ one is only allowed to conjugate by K. Then there are in general several K-conjugacy classes of θ-stable Cartans. There is a unique conjugacy class such that $\dim \mathfrak{t}$ (resp. $\dim \mathfrak{a}$) is maximal among all conjugacy classes. A Cartan in this classes is said to be a *maximally compact Cartan* (resp. *maximally split Cartan*).

When $\mathfrak{c}$ is changed from one conjugacy class to another, different types of roots in $\Phi_\mathfrak{c}$ will also change. heuristically, the bigger the $\mathfrak{t}$ part (resp. $\mathfrak{a}$ part), the more imaginary roots (resp. real roots) there will be.

Example 1.2. $G_0 = \mathbf{SL}(2,\mathbb{R})$, $(\mathfrak{g}, K) = (\mathfrak{sl}(2,\mathbb{C}), \mathbf{SO}(2,\mathbb{C}))$. In this case there are two conjugacy classes of Cartans: one is compact (i.e. $\mathfrak{c}_c = \mathfrak{t}_c \oplus 0$), the other is split (i.e. $\mathfrak{c}_s = 0 \oplus \mathfrak{a}_s$). For the compact Cartan, the only root is noncompact imaginary; for the split Cartan, the only root is real.

Example 1.3. $G_0 = \mathbf{SU}(2,1)$, $(\mathfrak{g}, K) = (\mathfrak{sl}(3,\mathbb{C}), \mathbf{GL}(2,\mathbb{C}))$. In this case there are two conjugacy classes of Cartans: the maximally compact Cartan $\mathfrak{c}_c = \mathfrak{t}_c \oplus 0$ is compact.

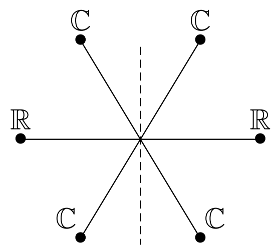
$\Phi_{\mathfrak{c}_c}$:



. (1.4)

θ fixes all roots. The maximally split Cartan $\mathfrak{c}' = \mathfrak{t}' \oplus \mathfrak{a}'$ is not split: $\dim \mathfrak{t}' = \dim \mathfrak{a}' = 1$.

$\Phi_{\mathfrak{c}'}$:

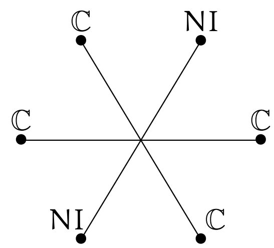


. (1.5)

θ acts by reflection by the dotted line.

Example 1.6. $G_0 = \mathbf{SL}(3, \mathbb{R})$, $(\mathfrak{g}, K) = (\mathfrak{sl}(3, \mathbb{C}), \mathbf{SO}(3, \mathbb{C}))$. In this case there are two conjugacy classes of Cartans. The maximally compact Cartan $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ is not compact.

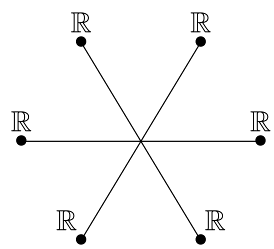
$\Phi_{\mathfrak{c}}$:



. (1.7)

θ acts by reflection across the NI-plane. The maximally split Cartan $\mathfrak{c}_s = 0 \oplus \mathfrak{a}_s$ is split:

$\Phi_{\mathfrak{c}_s}$:



. (1.8)

θ acts by -1.

1.2. K**-orbits in** X**.** How are these roots related to geometry? For each K-orbit Q, one can attach a K-conjugacy class of θ-stable Cartan $\operatorname{Ad} K \cdot \mathfrak{c}$ in the following way:

$$Q \ni x \rightsquigarrow \mathfrak{b}_x \supseteq \mathfrak{c} \rightsquigarrow \operatorname{Ad} K \cdot \mathfrak{c} \tag{1.9}$$

that is, take $x \in Q$, take the corresponding Borel $\mathfrak{b}_x$, take a θ-stable Cartan $\mathfrak{c}$ in $\mathfrak{b}_x$ (which can be done, see [Mil93, 5.3]), and then take the conjugacy class of $\mathfrak{c}$. Multiple orbits can be attached to the same Cartan $\mathfrak{c}$, and the preimage of $\mathfrak{c}$ is parameterized by the set of W_K-conjugacy classes of choices of negative roots in $\Phi_{\mathfrak{c}}$, where W_K is the Weyl group of K:

$$\begin{array}{ccc} W_K \backslash \left\{ \begin{array}{c} \text{choices of} \\ \Phi_{\mathfrak{c}}^- \subseteq \Phi_{\mathfrak{c}} \end{array} \right\} & \longrightarrow & \operatorname{Ad} K \cdot \mathfrak{c} \\ \downarrow & & \downarrow \\ K \backslash X & \longrightarrow & \operatorname{Ad} K \backslash \{\theta\text{-stable Cartans } \mathfrak{c} \subseteq \mathfrak{g}\} \end{array} . \tag{1.10}$$

For $x \in Q$, we have a involution θ_Q on $\mathfrak{h}^*$ and a set of negative roots $\Phi^- \subseteq \Phi$ obtained by pulling back θ along the natural map $\mathfrak{h}^* \xrightarrow{\sim} (\mathfrak{b}_x/\mathfrak{n}_x)^* \xrightarrow{\sim} \mathfrak{c}^*$. Write $\Phi^-_{Q,\mathbb{R}}$ for the set of Q-*real* (i.e. real with respect to θ_Q) roots in the set of negative roots determined by x in Q. In a similar way write $\Phi^-_{Q,\mathbb{C}}$, $\Phi^-_{Q,NI}$, $\Phi^-_{Q,CI}$.

Remark 1.11. Φ^- is determined on x up to G-conjugacy; the induced involution θ on $\mathfrak{h}^*$ is determined on $\mathfrak{c}$ up to K-conjugacy. So the pair (Φ^-, θ) is determined on x up to $G \cap K$-conjugacy, i.e. on the orbit $K \cdot x = Q$.

Knowing the root system data allows us to compute dimensions of the orbits. If $x \in Q$, then $\dim Q = \dim \mathfrak{k}/(\mathfrak{k} \cap \mathfrak{b}_x) = \dim \mathfrak{k} - \dim \mathfrak{k} \cap \mathfrak{b}_x$ which can be calculated by using root spaces. With some work one obtains

$$\dim Q = \frac{1}{2}\Big(|\Phi_{Q,CI}| + |\Phi_{Q,\mathbb{R}}| + \frac{1}{2}|\Phi_{Q,\mathbb{C}}| + |D_-(Q)|\Big) \tag{1.12}$$

where

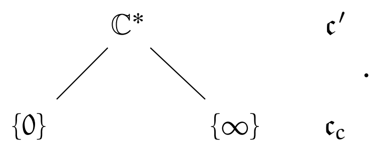
$$D_-(Q) = \left\{\alpha \in \Phi^-_{Q,\mathbb{C}} \mid \theta_Q\alpha \notin \Phi^-_Q\right\}. \tag{1.13}$$

Therefore, among all orbits attached to $\mathfrak{c}$, the maximal ones have maximal possible $|D_-(Q)|$, and the minimal ones have minimal possible $|D_-(Q)|$.

Definition 1.14. Given a θ-stable Cartan $\mathfrak{c}$, a **Zuckerman orbit** (resp. **Langlands orbit**) attached to $\mathfrak{c}$ is an orbit attached to $\mathfrak{c}$ with $D_-(Q) = \varnothing$ (resp. $D_-(Q) = \Phi^-_{Q,\mathbb{C}}$).

In particular Zuckerman (resp. Langlands) orbits are the minimal (maximal) ones attached to $\mathfrak{c}$. It is an easy combinatorial exercise that Zuckerman orbits and Langlands orbits always exist [Hec+, 5.1].

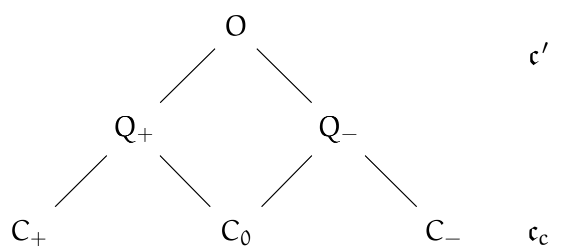
Example 1.15. $G_0 = \mathbf{SL}(2,\mathbb{R})$. We have three orbits: $\{0\}$, $\{\infty\}$, $\mathbb{C}^*$ in $X = \mathbb{P}^1$ attached to two conjugacy classes of Cartans $\mathfrak{c}_c$, $\mathfrak{c}'$:



(1.16)

If $\{\alpha, -\alpha\}$ are the roots, $\{0\}$ and $\{\infty\}$ corresponding to choosing α and $-\alpha$ as the negative root, respectively.

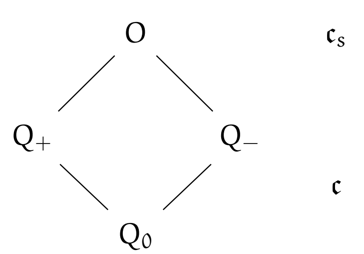
Example 1.17. $G_0 = \mathbf{SU}(2,1)$. There are six orbits:



(1.18)

where the three closed orbits are attached to $\mathfrak{c}_c$ and the other three attached to $\mathfrak{c}'$. $\Phi^-_{C_0}$ has two noncompact roots as simple roots, while one of the simple roots in $\Phi^-_{C_+}$ or $\Phi^-_{C_-}$ is compact. Both $\Phi^-_{Q_+}$ and $\Phi^-_{Q_-}$ have a real root as part of the simple roots, while the simple roots of Φ^-_O are both complex.

Example 1.19. $G_0 = \mathbf{SL}(3,\mathbb{R})$. There are four orbits:



(1.20)

where O is attached to $\mathfrak{c}_s$ and the other three attached to $\mathfrak{c}$. $\Phi^-_{Q_0}$ has two complex roots as simple roots, and $\Phi^-_{Q_+}$, $\Phi^-_{Q_-}$ both have one imaginary root as part of the simple roots.

2. Special cases

Slogan 2.1.

- *Existence of $\mathcal{I}(Q,\tau)$ is determined by the behavior of τ along all imaginary roots;*
- *Irreducibility of $\mathcal{I}(Q,\tau)$ is determined by the behavior of τ along complex and real roots.*

Let us examine irreducibility of $\mathcal{I}(Q,\tau)$ when there is only one type of root in Φ_Q.

2.1. **Imaginary roots only.** If there is only imaginary roots, then the corresponding Cartan $\mathfrak{c}$ is compact, $\mathfrak{c} = \mathfrak{t}$, and is a Cartan subalgebra of $\mathfrak{k}$. So $\mathfrak{k}$ is spanned by $\mathfrak{t}$ and compact imaginary root spaces. The stabilizer $\mathfrak{k} \cap \mathfrak{b}_x$ of a point is spanned by $\mathfrak{c} = \mathfrak{t}$ and negative compact imaginary root spaces in, which is a Borel in $\mathfrak{k}$. Therefore $Q \cong K/(K \cap B_x)$ is the flag variety of K and is closed in X.

By Kashiwara's equivalence, $\mathcal{I}(Q,\tau)$ is automatically irreducible. However existence of τ implies that λ must lift to a character of T.

Remark 2.2. In fact in this case $\mathcal{I}(Q,\tau)$ is a (limit of) discrete series.

2.2. **Complex roots only.** Suppose Q is attached to the Cartan $\mathfrak{c}$.

Fun Exercise 2.3. *Show that if $\Phi_{\mathfrak{c}}$ contains only complex roots, then the pair $(\mathfrak{g}, K)$ comes from a complex group, that is there is a complex group G_0 (viewed as a real group) such that $(\mathfrak{g}, K) = (\mathfrak{g}_0 \times \mathfrak{g}_0, \Delta G_0)$.*

Write X_0 for the flag variety of $\mathfrak{g}_0$. Then the categories

$$\mathrm{Mod}(\mathcal{D}_{X_0\times X_0,\lambda'}, \Delta G_0) \text{ and } \mathrm{Mod}(\mathcal{D}_{X_0,\lambda}, B_0) \tag{2.4}$$

are equivalent when $\lambda' \in (\mathfrak{h}_0 \times \mathfrak{h}_0)^*$ and $\lambda \in \mathfrak{h}_0^*$ are integral (by descent, see [BB85, 3.10]). Let's look at $\mathrm{Mod}(\mathcal{D}_{X_0,\lambda}, N_0)$ instead in order to allow non-integral λ's. Here we know exactly which standard modules are irreducible. Let C_w denote the Schubert cell of w. Then $\Gamma(X, \mathcal{I}(C_w, \lambda)) = M(w\lambda)^\vee$.

Irreducibility Criterion: Verma Modules 2.5. *Let λ be integrally antidominant with respect to roots in N_0. Then $M(w\lambda)$ is irreducible if and only if $w\lambda$ is integrally antidominant as well.*

Localizing and possibly translating to other chambers, we get

Irreducibility Criterion: Standard Highest Weight Modules 2.6. *Let*

$$\Phi_w^- = \left\{\alpha \in \Phi^- \mid w\alpha \notin \Phi^-\right\}, \tag{2.7}$$

$$\Phi_\lambda = \left\{\alpha \in \Phi \mid \alpha^\vee(\lambda) \in \mathbb{Z}\right\}. \tag{2.8}$$

$\mathcal{I}(C_w, \lambda)$ is irreducible if and only if $\Phi_w^- \cap \Phi_\lambda = \varnothing$.

2.3. **Real roots only.** Real roots produce new phenomenon that is not present in complex groups. Conceptually, the main part of the irreducibility criterion is the study of this special case.

In this case Q must be attached to a split Cartan $\mathfrak{c} = 0 \oplus \mathfrak{a}$, and by the dimension formula (1.12) Q is an open orbit. A typical example is

Example 2.9. $G_0 = \mathbf{SL}(2,\mathbb{R})$, $K = \mathbf{SO}(2,\mathbb{C}) = \mathbb{C}^*$ which acts on the open orbit $\mathbb{C}^* \subset \mathbb{P}^1$ by multiplicaiton by the square, i.e. $c_1 \cdot c_2 = c_1^2 c_2$ for $c_i \in \mathbb{C}^*$. The stabilizer of a point is

$$M = \{1, m\} \cong \mathbb{Z}/2. \tag{2.10}$$

There are two representations ε of M, the trivial representation $\varepsilon = 0$ and the sign representation $\varepsilon = -1$. They determine two K-equivariant connections τ_ε on $\mathbb{C}^*$ and hence two standard modules $\mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)$ on X.

$\mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)$ is reducible if and only if some sections can be extended to the closed orbits, or more precisely, the monodromy opertor around either $\{0\}$ or $\{\infty\}$ has eigenvalue 1. By explicit calculation, this condition can be seen equivalent to $\frac{1}{2}(\varepsilon + \alpha^\vee(\lambda)) \in \mathbb{Z}$ or $\frac{1}{2}(\varepsilon - \alpha^\vee(\lambda)) \in \mathbb{Z}$, or equivalently, $\alpha^\vee(\lambda) + \varepsilon \in 2\mathbb{Z}$. Therefore, $\mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)$ is irreducible if and only if

$$\alpha^\vee(\lambda + \rho) + \varepsilon \notin 2\mathbb{Z} + 1 \tag{2.11}$$

This is called the *parity condition*. This can be rephrased as

$$e^{\pi i \alpha^\vee(\lambda+\rho)} \neq -e^{\pi i \varepsilon}(m). \tag{2.12}$$

Remark 2.13. In the above example, the monodromic condition for $\{0\}$ and $\{\infty\}$ agree. This is special to linear groups. When considering non-linear covers of $\mathbf{PSL}(2,\mathbb{R})$ (for example the 3-fold cover of $\mathbf{PSL}(2,\mathbb{R})$) it can happen that sections of $\mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)$ extends to $\{0\}$ but not to $\{\infty\}$.

In general, the Lie algebra of the stabilizer of a point $x \in Q$ in K is trivial: $\mathfrak{k} \cap \mathfrak{b}_x = 0$ (this can be seen by considering root spaces). Therefore the stabilizer $T = K \cap B_x$ is a finite group. K-equivariant connections on Q are parameterized by representations of T.

Inside this T there are a bunch of elements m_α's analogous to the element $m \in M$ in the $\mathbf{SL}(2,\mathbb{R})$ case. For each negative (real) root α, take $\alpha^\vee \in \mathfrak{c}$, and let $m_\alpha = \exp(\pi i \alpha^\vee)$ which is either the identity or has order 2. Hence for any representation ω of $K \cap B_x$, $\omega(m_\alpha) = \pm 1$. The generalized parity condition with respect to α is

$$e^{\pi i \alpha^\vee(\lambda+\rho)} \neq -\omega(m_\alpha). \tag{2.14}$$

Then $\mathcal{I}(O, \tau)$ is irreducible if and only if the parity condition holds for all roots.

Question 2.15. *Is T generated by the m_α's? Abelian? Isomorphic to a product of $\mathbb{Z}/2$'s?*

Remark 2.16. For nonlinear G_0, one instead take the preimages of the m_α's in K. T can be non-abelian (this happens for the universal cover of $\mathbf{SL}(3,\mathbb{R})$, for example).

Example 2.17. $G_0 = \mathbf{SL}(3,\mathbb{R})$. K is the subgroup of orthogonal matrices. The split Cartan subgroup C is the diagonal subgroup, and the Borel subgroup B_x of upper-triangular matrices corresponds to a point x in the open orbit O. In this case

$$T = K \cap B_x = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, m_\alpha = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, m_\beta = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, m_\gamma = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}. \tag{2.18}$$

3. Statement of Criterion

In this section we state the precise criterion for irreducibility of $\mathcal{I}(Q,\tau)$. Let $x \in Q$ and let $\mathfrak{c} \subset \mathfrak{b}_x$ be a θ-stable Cartan to which Q is attached. Let $T \subset K \cap B_x$ corresponding to $\mathfrak{t}$. Recall that, for each real root in Φ_Q^-, there is an element $m_\alpha \in T$ defined by

$$m_\alpha = \exp(\pi i \alpha^\vee) \tag{3.1}$$

where $\alpha^\vee \in \mathfrak{c}$.

Irreducibility Criterion: Zuckerman orbits 3.2. *$\mathcal{I}(Q,\tau)$ is irreducible if and only if*

- *For each $\alpha \in \Phi^-_{Q,\mathbb{R}}$, $e^{\pi i \alpha^\vee(\lambda+\rho)} \neq -\tau(x)(m_\alpha)$.*

Here $\tau(x)$ is the geometric fiber of τ at x, viewed as a representation of $K \cap B_x$.

The general parity condition involves complex roots. Recall that

$$D_-(Q) = \{\alpha \in \Phi^-_{Q,\mathbb{C}} \mid \theta_Q \alpha \notin \Phi^-\} \tag{3.3}$$

$$= \{\alpha \in \Phi^-_{Q,\mathbb{C}} \mid -\theta_Q \alpha \in \Phi^-_{Q,\mathbb{C}}\} \tag{3.4}$$

The $-\theta_Q$-orbits in $D_-(Q)$ consists of pairs $\{\alpha, -\theta_Q\alpha\}$. Let $C \subset G$ be the Cartan subgroup of $\mathfrak{c}$.

Definition 3.5. Let A be a set of representatives of $-\theta_Q$-orbits in $D_-(Q)$. For $\beta \in A$, let e^β be the corresponding character of C.

For each real root $\alpha \in \Phi_Q^-$, we say that τ satisfies the **$\mathbf{SL}_2$-parity condition with respect to** α if

$$e^{\pi i \alpha^\vee(\lambda+\rho)} \neq -\tau(x)(m_\alpha) \prod_{\beta \in A} e^\beta(m_\alpha). \tag{3.6}$$

Irreducibility Criterion 3.7. *([Hec+, 8.8]) $\mathcal{I}(Q,\tau)$ is irreducible if and only if*

- *$D_-(Q) \cap \Phi_\lambda = \varnothing$, and*

- *For each $\alpha \in \Phi^-_{Q,\mathbb{R}}$, τ satisfies the $\mathbf{SL}_2$-parity condition with respect to α.*

On Zuckerman orbits, $D_-(Q) = \varnothing$, so we recover the criterion 3.2.

Here the condition $D_-(Q) \cap \Phi_\lambda = \varnothing$ should be thought of the analogous of the condition $\Phi^-_w \cap \Phi_\lambda = \varnothing$ for irreducibility of Verma modules. In fact, if G_0 is complex, $D_-(C_w) = \Phi^-_w$. The extra bit in the definition of the parity condition is formulated so that the statement is uniform for all orbits. It is the translation of the parity condition by simple reflections. The precise meaning of "translation" will be made clear in the next section.

Example 3.8. $G_0 = \mathbf{SL}(3,\mathbb{R})$, $x \in Q = O =$ the open orbit. As computed in Example 2.17, the stabilizer is $K \cap B_x = \{1, m_\alpha, m_\beta, m_\gamma\} \cong (\mathbb{Z}/2)^2$ where $m_\gamma = m_\alpha m_\beta$. It has four irreducible representations

$$\omega_{trv,trv}, \quad \omega_{trv,sgn}, \quad \omega_{sgn,trv}, \quad \omega_{sgn,sgn}, \tag{3.9}$$

each one of those gives rise to a connection τ that is compatible with any λ (because $K \cap B_x$ is discrete).

Take $\tau_{trv,sgn}$ corresponding to $\omega_{trv,sgn}$. In order for $\mathcal{I}(O, \tau_{trv,sgn})$ to be irreducible, we need

$$e^{\pi i \alpha^\vee(\lambda+\rho)} \neq -\omega_{trv,sgn}(m_\alpha) = -1 \implies \alpha^\vee(\lambda) \notin 2\mathbb{Z}; \tag{3.10}$$

$$e^{\pi i \beta^\vee(\lambda+\rho)} \neq -\omega_{trv,sgn}(m_\beta) = 1 \implies \beta^\vee(\lambda) \notin 2\mathbb{Z}+1; \tag{3.11}$$

$$e^{\pi i \gamma^\vee(\lambda+\rho)} \neq -\omega_{trv,sgn}(m_\gamma) = 1 \implies \gamma^\vee(\lambda) \notin 2\mathbb{Z}. \tag{3.12}$$

(3.13)

Other choices representations can be analyzed similarly.

Example 3.14. $G_0 = \mathbf{SU}(2,1)$. Standard modules on closed orbits are clearly irreducible (if they exist).

The other three orbits are attached to the maximally split Cartan

$$\mathfrak{c}' = \left\{ \begin{pmatrix} a & & b \\ & -2a & \\ b & & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}. \tag{3.15}$$

The orbit Q_+ is determined by the roots

$$\alpha = 3a - b \ (\mathbb{C}) \tag{3.16}$$

$$\beta = 2b \ (\mathbb{R}) \tag{3.17}$$

$$\gamma = 3a + b \ (\mathbb{C}). \tag{3.18}$$

If $x \in Q_+$ so that $\mathfrak{b}_x \supseteq \mathfrak{c}$, then after some calculation one can see that

$$K \cap B_x = \left\{ \begin{pmatrix} z & bz & \\ & z^{-2} & \\ & & z \end{pmatrix} \mid z \in \mathbb{C}^*, b \in \mathbb{C} \right\} \cong \mathbb{C}^* \ltimes \mathbb{C} \tag{3.19}$$

whose irreducible representations are just those of $\mathbb{C}^*$, which are parameterized by non-negative integers $n \in \mathbb{Z}$. Write τ_n for the corresponding connection. In order for $\mathcal{I}(Q_+, \tau)$ to be a $\mathcal{D}_{X,\lambda}$-module, one needs

$$\lambda \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} = n. \tag{3.20}$$

The only real root is β with

$$\beta^\vee = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \in \mathfrak{c} \text{ and } m_\beta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{3.21}$$

The parity condition says

$$e^{\pi i \beta^\vee(\lambda+\rho)} \neq -\tau_n(m_\beta) = -(-1)^n = \begin{cases} -1 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}. \tag{3.22}$$

Here $\rho = -\gamma = -(3a+b)$, so $\beta^\vee(\rho) = -1$. The above condition is equivalent to

$$\lambda \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \text{ is either not an integer, or its parity is different than } \lambda \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix}. \tag{3.23}$$

This can be rewritten as

$$\text{Either } \frac{1}{2}\lambda \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} - \frac{1}{2}\lambda \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \notin \mathbb{Z}, \text{ or } \lambda \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \notin \mathbb{Z}. \tag{3.24}$$

But $\alpha^\vee = \frac{1}{2}\left(\begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}\right)$. So the above condition is equivalent to either $\alpha^\vee(\lambda) \notin \mathbb{Z}$ or $\beta^\vee(\lambda) \notin \mathbb{Z}$, i.e. to non-integrality of λ.

The orbit Q_- can be analyzed in a similar way.

Now consider the open orbit O. Then

$$K \cap B_x = \left\{ \begin{pmatrix} z & & \\ & z^{-2} & \\ & & z \end{pmatrix} \mid z \in \mathbb{C}^* \right\} = T. \tag{3.25}$$

Connections on O are again parameterized by $n \in \mathbb{Z}$, with the same requirement (3.20). The parity condition says

$$e^{\pi i \beta^\vee(\lambda+\rho)} \neq -\tau_n(m_\beta) e^{-\alpha}(m_\beta) = (-1)^n. \tag{3.26}$$

Here $\rho = -\beta$, so $\beta^\vee(\rho) = -2$. The parity condition is then equivalent to (3.23). The other condition $D_-(O) \cap \Phi_\lambda = \{-\alpha, \gamma\} \cap \Phi_\lambda = \varnothing$ is subsumed by (3.23).

Proposition 3.27. *Let Q be a K-orbit in the case of $G_0 = \mathbf{SU}(2,1)$. Then*

- *Standard modules on closed orbits are irreducible (whenever they exist);*
- *Standard modules on non-closed orbits are irreducible if and only if λ is not integral.*

4. Outline of proof

4.1. Main tool: intertwining functors.

Definition 4.1. For $w \in W$, let Z_w denote the G-orbit in $X \times X$ corresponding to w, with projection maps

$$X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X. \tag{4.2}$$

The **intertwining functor** is

$$\mathbb{I}_w : D^b(\mathcal{D}_{X,\lambda}) \to D^b(\mathcal{D}_{X,w\cdot\lambda}), \quad \mathcal{F} \mapsto p_{1*}\left(p_1^* \mathcal{L}_{w\rho-\rho} \underset{\mathcal{O}_{Z_w}}{\otimes} p_2^* \mathcal{F}\right). \tag{4.3}$$

See [DV] for a definition using convolution.

Theorem 4.4 ([Mil, Section 3.3])**.**

- *$\mathbb{I}_w$ is the left derived functor of $H^0\mathbb{I}_w$.*
- *$\mathbb{I}_w$ is an equivalence of derived categories.*
- *If $\Phi_w^- \cap \Phi_\lambda = \varnothing$, then*

$$H^0\mathbb{I}_w : \mathrm{Mod}(\mathcal{D}_{X,\lambda}) \to \mathrm{Mod}(\mathcal{D}_{X,w\cdot\lambda}) \tag{4.5}$$

is an equivalence of abelian categories with inverse $H^0\mathbb{I}_{w^{-1}}$.

In the framework of localization, we know there is a geometric realization of $\mathcal{U}(\mathfrak{g})/\lambda$ for each Weyl group translate of λ. Intertwining functors is a geometric construction that allows one to move from one realization to another.

The main use of intertwining functors is their ability to increase the dimension of supports without loosing too much information.

4.2. Reduction to Zuckerman orbits.

The first usage of intertwining functors is the reduction to Zuckerman orbits, i.e. reducing 3.7 to 3.2 [Hec+, 8.5]. The idea is exactly the same as Verma module case.

Geometric proof of 2.6. We prove the irreducibility criterion for standard highest weight modules using intertwining functors.

First, the module at a point $\mathcal{I}(C_1, w\cdot\lambda)$ is certainly irreducible. Then, one shows that

$$H^0\mathbb{I}_{w^{-1}}\mathcal{I}(C_1, w\cdot\lambda) = \mathcal{I}(C_w, \lambda). \tag{4.6}$$

To see this, look at the following diagram

$$\begin{array}{ccccc} & & C_w & \xrightarrow{p_2|_{C_w}} & C_1 \\ & {\scriptstyle j_w}\swarrow & \downarrow & & \downarrow{\scriptstyle j_1} \\ X & \xleftarrow[p_1]{} & Z_{w^{-1}} & \xrightarrow{p_2} & X \end{array} \tag{4.7}$$

Here, by definition of $Z_{w^{-1}}$, $C_w = p_2^{-1}(C_1)$, i.e. the right square is Cartesian. Also, the arrow $C_w \to X$ is the same as the usual inclusion. Hence by base change and definition of $\mathbb{I}_{w^{-1}}$,

$$\mathbb{I}_{w^{-1}}\mathcal{I}(C_1, w\cdot\lambda) = \mathcal{L} \underset{\mathcal{O}_X}{\otimes} p_{1*}p_2^*j_{1*}\tau \tag{4.8}$$

$$= \mathcal{L} \underset{\mathcal{O}_X}{\otimes} j_{w*}(p_2|_{C_w})^*\tau \tag{4.9}$$

$$= \mathcal{I}(C_w, \lambda). \tag{4.10}$$

Finally, if $\Phi_w^- \cap \Phi_\lambda = \varnothing$, then $H^0\mathbb{I}_w$ is an equivalence of abelian categories $\mathrm{Mod}(\mathcal{D}_{X,\lambda}, N) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{D}_{X,w\cdot\lambda}, N)$. Its inverse $H^0\mathbb{I}_{w^{-1}}$ is also an equivalence. Thereefore $\mathcal{I}(C_w, \lambda)$ is irreducible by irreducibility of $\mathcal{I}(C_1, w \cdot \lambda)$.

The other direction is more involved and proceeds by induction on $\ell(w)$. Write $w = w's_\alpha$. If α is integral, one analyze $H^0\mathbb{I}_{s_\alpha}$ and $H^1\mathbb{I}_{s_\alpha}$ carefully and find a nonzero proper submodule in $\mathcal{I}(C_w, \lambda)$. Hence α is forced to be non-integral to λ, $H^0\mathbb{I}_{s_\alpha}$ is an equivalence and we are reduced to the case for w' with smaller length. ∎

The exact same idea works in the Harish-Chandra setting. For any orbit Q, one can find a Zuckerman orbit Q^Z attached to the same Cartan and a Weyl group element w such that we have the same diagram

$$\begin{array}{ccccc} & & Q & \xrightarrow{p_2|_Q} & Q^Z \\ & {\scriptstyle j_Q}\swarrow & \downarrow & & \downarrow{\scriptstyle j_{Q^Z}} \\ X & \xleftarrow[p_1]{} & Z_{w^{-1}} & \xrightarrow{p_2} & X \end{array} \tag{4.11}$$

with the property that the right square is Cartesian and the map $Q \to X$ is the usual inclusion. The precise condition for w is

$$\Phi_w^- \cap (-\theta_Q\Phi_w^-) = \varnothing \text{ and } D_-(Q) = \Phi_w^- \cup (-\theta_Q\Phi_w^-). \tag{4.12}$$

From this one can also show that $\Phi_w^- \cap \Phi_\lambda = \varnothing$ is equivalent to $D_-(Q) \cap \Phi_\lambda = \varnothing$. Therefore the same strategy goes through (although one needs to replace the analysis of integral simple intertwining functor $\mathbb{I}_{s_\alpha}$ by something else).

Assuming 3.2, then one can obtain a criterion for general orbits by translating the $\mathbf{SL}_2$-parity condition along intertwining functors. The extra part in the $\mathbf{SL}_2$-parity condition for general orbits is the result of translation.

4.3. **Reduction to split pairs.** To prove the result on Zuckerman orbits, we want to reduce to open orbits for split pairs, which we had considered in §2.3.

A special property of Zuckerman orbits Q^Z is that the set of negative roots $\Phi_{Q^Z}^-$ it corresponds can be extended to a parabolic set of roots given by $\Phi_{Q^Z}^- \cup \Phi_{Q^Z,\mathbb{R}}$, which corresponds to the subset Θ of simple roots consisting of real simple roots. If $\pi_\Theta : X \to X_\Theta$ is the projection to the corresponding partial flag variety, then $\pi_\Theta(Q^Z)$ is a closed orbit. So the preimage

$$\pi_\Theta^{-1}(\pi_\Theta(Q^Z)) \subset X \tag{4.13}$$

is a smooth closed subvariety. By Kashiwara's equivalence, irreducibility of $\mathcal{I}(Q^Z, \tau)$ can be analyzed in the category of $\mathcal{D}_{\pi_\Theta^{-1}(\pi_\Theta(Q^Z)),\lambda}$-modules.

Let $y \in \pi_\Theta(Q^Z)$. Since we are over a K-orbit $\pi_\Theta(Q^Z)$, descent says that pulling back to $\pi^{-1}(y)$ is an equivalence of categories of equivariant $\mathcal{D}$-modules:

$$\mathrm{Mod}(\mathcal{D}_{\pi_\Theta^{-1}(\pi_\Theta(Q^Z)),\lambda}, K) \cong \mathrm{Mod}(\mathcal{D}_{\pi^{-1}(y),\lambda}, \mathrm{Levi}(K \cap P_y)) \tag{4.14}$$

in the same way that K-equivariant $\mathcal{D}$-modules on an orbit are equivalent to representations of the stabilizer of a point. But $\pi^{-1}(y)$ is the flag variety of $[\mathfrak{l}_y, \mathfrak{l}_y]$ (here $\mathfrak{l}_y$ is the Levi of $\mathfrak{p}_y$), the latter category is the category of Harish-Chandra modules for the split pair $([\mathfrak{l}_y, \mathfrak{l}_y], \mathrm{Levi}(K \cap P_y))$, and our module $\mathcal{I}(Q^Z, \tau)$ becomes the standard module on the open orbit in $\pi^{-1}(y)$. So we are reduced to studying standard modules on the open orbit for split pairs.

Example 4.15. $G_0 = \mathbf{SU}(2,1)$. If we consider the Zuckerman orbit Q_+, then Θ is the real simple root, $\pi_\Theta^{-1}(\pi_\Theta(Q_+)) = Q_+ \cup C_+ \cup C_0$, and $\pi_\Theta^{-1}(y) \cong \mathbb{P}^1$ on which $\mathrm{Levi}(K \cap P_y)$ acts with three orbits. $\mathcal{I}(Q_+, \tau)$ becomes one of the standard modules on the open orbit in $\mathbb{P}^1$. So we are reduced to $\mathbf{SL}(2, \mathbb{R})$ calculation.

4.4. **Proof for split pairs.**

Lemma 4.16. *Let* $(\mathfrak{g}, K)$ *be split. Let* O *be the open* K*-orbit in* X *and let* τ *be a* K*-equivariant connection on* O*. Then* $\mathcal{I}(O, \tau)$ *is irreducible if and only if* τ *satisfies the* $\mathbf{SL}_2$*-parity condition for all roots* α*.*

Idea of proof. Suppose $\mathcal{I}(O, \tau)$ is not irreducible. Then it has a irreducible quotient $\mathcal{K}$. Its support is necessarily irreducible, and is hence the closure of a K-orbit.

First consider the case where $\mathrm{Supp}\,\mathcal{K}$ has codimension 1 in X. Then we can detect this quotient in a $\mathbf{SL}(2, \mathbb{R})$ fiber. In more detail, for a simple root α, write $\pi_\alpha : X \to X_\alpha$ for the projection to the partial flag variety. There exists a choice of α transversal to $\mathrm{Supp}\,\mathcal{K}$, i.e. so that $\pi_\alpha^{-1}(\pi_\alpha(\mathrm{Supp}\,\mathcal{K}))$ is dense in X. Any fiber $\pi_\alpha^{-1}(y)$ of π_α is $\mathbb{P}^1$ and the induced action of $\mathrm{Levi}(K \cap P_y)$ looks like the $\mathbf{SL}(2, \mathbb{R})$ or $\mathbf{PSL}(2, \mathbb{R})$-situation. After restriction to the fiber, $\mathcal{K}|_{\pi_\alpha^{-1}(y)}$ is supported on the closed orbits, and $\mathcal{I}(O, \tau)|_{\pi_\alpha^{-1}(y)}$ is a standard module on the open orbit that contains $\mathcal{K}|_{\pi_\alpha^{-1}(y)}$ as a quotient. So the parity condition for $\mathcal{I}(O, \tau)|_{\pi_\alpha^{-1}(y)}$ must fail, which contradicts our assumption. Hence such $\mathcal{K}$ cannot exist.

If $\mathrm{Supp}\,\mathcal{K}$ has codimension greater than 1, then there is a $w \in W$ so that $H^0\mathbb{I}_w \mathrm{Supp}\,\mathcal{K}$ has codimension 1 in X. If the $\mathbf{SL}_2$-parity condition holds for all roots, then for each s_α appearing in a reduced expression of w, $H^0\mathbb{I}_{s_\alpha}\mathcal{I}(O, \tau) = \mathcal{I}(O, \tau')$ for some connection τ', and τ' also satisfies all parity conditions (this can be proven by reducing to the $\mathbf{SL}(2, \mathbb{R})$ case where the intertwining functor can be calculated explicitly). Hence $H^0\mathbb{I}_w\mathcal{I}(O, \tau) = \mathcal{I}(O, \tau_w)$ for some connection τ_w. As $H^0\mathbb{I}_w$ is right exact, we see that $H^0\mathbb{I}_w\mathcal{K}$ is a nonzero quotient of $\mathcal{I}(O, \tau_w)$ supported in a codimension 1 subset and we are reduced to the previous case. ∎

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