24 Mar 2022 Kari. V

Fourier transform

e^{λ}: D-mod on $\mathbb{C}$

V, V^*,

$$\begin{tikzcd} & V \otimes V^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[r, "c"] & \mathbb{C} \\ V & & V^* \end{tikzcd}$$

/étale, e^{λ} = Artin - Schreier.

$$\hat{\mathcal{F}} := Rp_{2!}\left(p_1^*\mathcal{F} \otimes c^*(e^{\lambda})\right)$$

Topological (Fourier - Sato): : $\mathbb{R}_+$ conic.

Work / $\mathbb{R}$.

$$\begin{tikzcd} & V \times V^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & Z = \{ (x, \xi) \mid \xi(x) \geq 0 \} \arrow[l, hook', "i"'] \\ V & & V^* \end{tikzcd}$$

$$\hat{\mathcal{F}} = Rp_{2!}\, i_* i^! p_1^* \mathcal{F}.$$

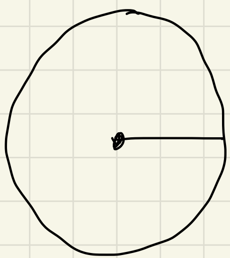
Nearby cycles:

Topologically:

$$\begin{array}{c} \mathfrak{X} \\ f \downarrow \\ \mathbb{C}. \end{array}$$

$\mathcal{F}$ a sheaf on $\mathfrak{X} - f^{-1}(0)$.

$$\Psi_f \mathcal{F} := i^* R\tilde{j}_* \mathcal{F}\big|_{\mathfrak{X}^{\geq 0}}$$



$$\begin{tikzcd} \mathfrak{X}^0 \arrow[r, hook, "i"] \arrow[d] & \mathfrak{X}^{\geq 0} \arrow[d] & \mathfrak{X}^{>0} \arrow[l, hook', "\tilde{j}"'] \arrow[d] \\ \mathbb{R}^0 \arrow[r, hook] & \mathbb{R}^{\geq 0} & \mathbb{R}^{>0} \arrow[l, hook'] \end{tikzcd}$$

Move the cut.

get monodromy $\mu \circlearrowright \Psi_f$.

No cut take universal cover of $\mathbb{C}^*$ ← works in all kinds of settings.

Ting: $\mathfrak{g}_1 \xrightarrow{f} \mathfrak{g}_1 /\!/ K = \mathfrak{a}/W_a$ & higher dim.

$\mathfrak{a}^{r.s.}/W_a$ Monodromy by BW_a.

This works ~~if~~ because f sat. A_f.

Higer dim'l nearby cycles complicated:

Deligne: vanishing topos.

Last time

rep. theory / geometry of real groups.

Both real & complex picture.

1. My Park city notes.
2. Paper with Dougal (beginning)

3. Our website Semester 2, 2019.

Talks July 30, Nov 5, Nov 12 (Introd. to Langlands).

$\underbrace{\text{July 30}}_{\text{general}}$, $\underbrace{\text{Nov 5, Nov 12}}_{\text{local langlands for tori.}}$

4. Semester 2, 2020, set up of geometric story.

Langlands parameters:

$W_{\mathbb{R}}$ = Weil group / $\mathbb{R}$.

The unique non-split extension

$$0 \to \mathbb{C}^* \to W_{\mathbb{R}} \to \Gamma \to 0.$$

$$\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$$

Γ acts on $\mathbb{C}^*$ by complex conjugation.

$W_{\mathbb{R}}$ is not the semi-direct product.

- The fundamental class of class field theory $\in H^2(\Gamma, \mathbb{C}^*)$.

Form $H^1(W_{\mathbb{R}}, \check{G})$. $\quad$ G: complex reductive

$\check{G}$ dual group

Langlands par.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d] & W_{\mathbb{R}} \arrow[r] \arrow[d, "\varphi"] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \check{G} \arrow[r] & {}^L G \arrow[r] & \Gamma \arrow[r] & 1
\end{tikzcd}$$

not algebraic

(Γ acts by outer auto on $\check{G}$).

$\Longrightarrow$ ${}^L G = \check{G} \rtimes \Gamma$

fix an outer autom.

$\varphi|_{\mathbb{C}^*}$ maps to s.s. elts. $\quad$ consider this under conj. by $\check{G}$.

Langlands parameters on $\check{G}$ side $\rightsquigarrow$ repr. of real forms of G.

$j \in W_{\mathbb{R}}$. $\quad j^2 = -1$. $\quad j \rightsquigarrow$ non-trivial elt in Γ.

j acts by complex conj. on $\mathbb{C}^*$.

a) $\varphi(j) \in {}^LG - \check{G}$.

b) $\varphi(\mathbb{C}^*) \subseteq \check{G}_{s.s.}$ $\qquad \lambda, \mu \in \mathfrak{g}^{s.s.} \quad [\lambda, \mu] = 0$

$$\varphi|_{\mathbb{C}^*} : \varphi(e^t) = \exp(t\lambda + \bar{t}\mu) \qquad (z^\lambda \cdot \bar{z}^\mu)$$

$$\varphi(e^{2\pi i}) = \varphi(1) = 1 \Rightarrow \exp(2\pi i \lambda) = \exp(2\pi i \mu).$$

$$\rightsquigarrow \mathrm{Ad}\,\varphi(j)\,\lambda = \mu.$$

The data of the Langlands parameters:

$$\check{G} \text{ acts } \left\{ \begin{array}{l} y = \exp(\pi i \lambda)\,\varphi(j) \in {}^LG - \check{G} \\ \lambda \in \check{\mathfrak{g}}^{s.s} \end{array} \middle| \begin{array}{l} y^2 = \exp(2\pi i \lambda) \\ [\lambda, \mathrm{Ad}(y)\lambda] = 0. \end{array} \right\}$$

Fix λ, $\quad Z_{\check{G}}(\lambda)$ acts on the y's.

$$\operatorname{Im}\varphi \subseteq \underbrace{Z_{\check{G}}(\exp 2\pi i \lambda)}_{\text{endoscopic gp}}, \quad \text{replace } \check{G} \text{ by}$$

Assume $\exp(2\pi i \lambda) \in Z(\check{G})$.

$$\{(y,\lambda)\}/\check{G} \longrightarrow \{ y \in {}^{L}G - \check{G} \mid y^2 \in Z(\check{G}) \} \qquad (1)$$

$$\{(y,\lambda)\}/\check{G} \longrightarrow \{ \lambda \in \check{\mathfrak{g}}_{ss}/\!/\check{G} = \check{\mathfrak{h}}/\check{W} = \mathfrak{h}^*/W \} \qquad (2)$$

(1) Set of strong involutions on $\check{G}$.

$y \rightsquigarrow \theta_y = \operatorname{Ad}(y)$, $\theta_y^2 = id$.

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② Inf'l char on the dual side.

Assume λ regular $\Rightarrow Z_{\check{\mathfrak{g}}}(\lambda) = \check{\mathfrak{t}}$.

<u>Comment</u>: Write $\check{k}_y = \check{\mathfrak{g}}^{\check{\theta}_y}$.

$[\lambda, \mathrm{Ad}(y)(\lambda)] = 0 \Rightarrow [\check{T}, \theta_y \check{T}] = 1$

$\check{\theta}_y : \check{T} \circlearrowleft$

Fix y, fix the image of λ in $\check{\mathfrak{h}}/\check{W}$.

$\check{k}_y = \check{\mathfrak{g}}^y = ?$ $\qquad y^{-1} g y = g \Rightarrow g \in \check{k}_y$.

Let $\check{k}_y$ act on the data after fixing the image of λ in $\check{\mathfrak{h}}/\check{W}$.

$$\begin{tikzcd}
 & \check{\mathfrak g} \arrow[r] & \check{\mathfrak g}/\!/\check G = \check{\mathfrak h}/\!/\check W \\
 & \tilde{\check{\mathfrak g}} \arrow[u] \arrow[r] \arrow[dl] & \check{\mathfrak h} \arrow[u] \\
\check{\mathcal B} & &
\end{tikzcd}$$

$$\tilde{\check{\mathfrak g}} = \{(x, \check B) \mid x \in \check B\}$$

The x's give $\check T \le \check B$, if the image is fixed in $\check{\mathfrak h}$
then consider such pairs.

Now the $\check T$ has to by θ_y-stable, finitely many such pairs under $\check K_y$-conj.

<u>Claim</u>: These points in the fiber up to $\check K_y$ conj
$= \check{\mathcal B}/\check K_y$.

Langlands param/ $\check{G}$ $= \coprod_{y \text{ strong real form}} \check{B}/\check{K}_y$.