Rep. theory 2022.

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Research seminars. org

1. Atlas of Lie gps
2. Automorphic seminar. (IHES summer school).

classical view of reps theory:

G top. gp.

1. Understand the small/irred. reps of G.
2. G acts X. Fun (X) $\leadsto$ decomposition into irreducibles.
3. Understand the cat of reps.

1 + 3 for real gps.

G real gp. Semi-alg (conn. component / $\mathbb{R}$).

$\mathbb{R}^*$, $\mathbb{R}^+$ (not alg gp)

- All cpt connected Lie gps are algebraic.

G reductive, alg gp$/\mathbb{C}$.

$$\updownarrow$$

Root datum $T \subseteq G$ max torus.

$$(X^*(T), \underset{\text{root}}{\Phi}, X_*(T), \underset{\text{coroot}}{\Phi^\vee}).$$

A real form $\sigma : G \longrightarrow G$ anti-involution.

$$G_{\mathbb{R}} := G^\sigma.$$

Pick $K_{\mathbb{R}} \subseteq G_{\mathbb{R}}$ maxil cpct.

$\theta : G_{\mathbb{R}} \circlearrowright$ inv. $/\mathbb{R}$. $\quad G_{\mathbb{R}}^\theta = K_{\mathbb{R}}$.

$\Downarrow$ $\theta : G \to G.$ $\qquad G^\theta = K.$

$\sigma \longleftrightarrow \theta$: 1 to 1. up to conj. by G.

$G = SL(2,\mathbb{C}) \supseteq SL(2,\mathbb{R}) = G_{\mathbb{R}}$ $\qquad \boxed{\theta A = {}^t A^{-1}}$

$\cup \qquad\qquad \cup$

$K = SO(2,\mathbb{C}) \supseteq SO(2,\mathbb{R}) = K_{\mathbb{R}}$

$SL(2,\mathbb{C}) \supseteq SU(2)$ $\qquad \theta = id$

$\| \qquad\qquad \|$

$SL(2,\mathbb{C}) \supseteq SU(2)$

To underst real form: Galois cohomology $\Gamma = Gal(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}.$

(*) It is more natural to use strong inner forms.

$\longrightarrow$ Adam-du Cloux: Algorithms ... ...

<u>Repsentations:</u>

$G_{\mathbb{R}}$, V top. v.s $/\mathbb{C}$.

local.

CH LC

complete

complete Hausdorff

$G_{\mathbb{R}} \times V \longrightarrow V$ contions. &

$g \in G_{\mathbb{R}}$ $V \to V$ linear.

If V Hilbert space & $G_{\mathbb{R}}$ acts via Unitary operators, we call this rep unitary.

$\mathfrak{g}_{\mathbb{R}}$ acts V^{∞}, smooth vectors

$\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$

$V^{\infty} = \{ v \in V, \; G_{\mathbb{R}} \to V, \; g \mapsto g.v$ is smooth $\}$

$V^{\infty} \supseteq V_{K_{\mathbb{R}}} = K_{\mathbb{R}}$- finite vectors, $K_{\mathbb{R}} \subseteq G_{\mathbb{R}}$ max cpt.

$V_{K_{\mathbb{R}}} = V_K = (\mathfrak{g}, K)$ - mod.

The reps is <u>admissible</u> if $\underbrace{V_K = \bigoplus_{\pi \in \hat{K}} V_\pi^{\oplus n_\pi}}_{\text{smooth}}, \; n_\pi < \infty$.

These things are dense in V;

$$\{\text{Admissible reps of } G_{\mathbb{R}}\} \;\underset{M}{\overset{m}{\leftleftarrows}}\!\!\longrightarrow \{(\mathfrak{g},K)\text{-mod}\} \quad \text{exact}$$

Maximal & minimal globalizations

m = analytic functions

M = hyper functions

Geometry:

$\mathcal{B} = G/B$ flag mfd. $\qquad N \subseteq B$ unip. radical

$$\begin{array}{c} \mathcal{B} = G/B \\ \uparrow \\ \widetilde{\mathcal{B}} = G/N \end{array} \qquad H\text{-fibration}$$

$\lambda \in \mathfrak{h}^*$, $\quad \mathcal{D}_\lambda$: twisted ring of diff. operators on G/B

- If λ regular dom't.

$$\{ \mathcal{D}_\lambda\text{-mods} \} \xrightarrow[\text{Serre}]{\Gamma} \{ U(\mathfrak{g})_\lambda\text{-mod} \}$$

- If λ on the wall, get a quotient. $RHS = LHS/\sim$

Apply this to $(\mathfrak{g}, K)$-mods.

All K's will be conjugat

<u>Then</u>: λ: regular.

$\mathcal{D}_\lambda$-mods will be holonomic & regular.

$\lambda \in \mathfrak{h}^*$, $\rightsquigarrow \chi_\lambda \in \operatorname{Spec} Z(\mathfrak{g})$.

$$
\begin{tikzcd}
\{G_{\mathbb{R}}\text{-reps}\} \arrow[r, shift left, "\text{admissible}"] & \{(\mathfrak{g},K)\text{-mods}\} \arrow[l, shift left] \\
 & \{K\text{-equiv Coh-}\mathcal{D}_{\lambda}\text{-mods reg. hol}\} \arrow[u, "r"'] \arrow[d, "\cong"', "RH"] \\
 & P_K(\mathcal{B})_{-\lambda} \arrow[dl, dashed] \arrow[d, phantom, "\cap"] \\
D_{G_{\mathbb{R}}}(\mathcal{B})_{-\lambda} \arrow[r, leftrightarrow, "\cong"', "\text{Matsuki for sheaves}"] \arrow[uuu, dashed, "\text{Kashiwara}"] & D_K(\mathcal{B})_{-\lambda}
\end{tikzcd}
$$

Take derived cat of $G_{\mathbb{R}}$-reps

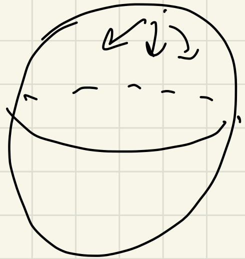
algeb.

not algebraic

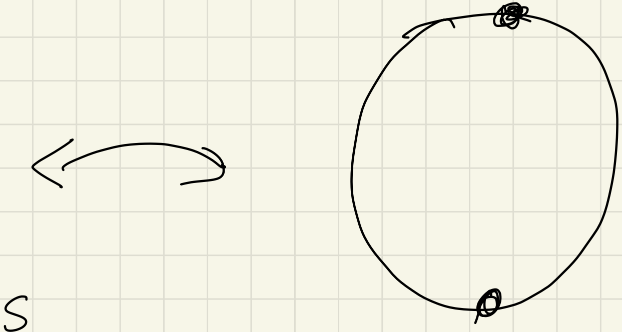
up to integer

① MUV : Equiv. of cats

$SL(2,\mathbb{R})$ $\qquad \mathcal{B} = \mathbb{C}P^1$



$SL(2,\mathbb{R})$-orbits



$K = SO(2,\mathbb{C})$ - orbits

② Ionov-Yun: Matsuki is Ringel duality.

Questions

① Which reps are unitary? ($\exists$ as alg).

(Orbit method) coadjoint orbits.

Thm [H-C]
Unitary reps are admissible. $\boxtimes$

H-C: An irred rep is unitary iff its $(\mathfrak{g}, K)$-mod preserves a positive def. $\mathfrak{g}_{\mathbb{R}}$-inv Hermitian form.

2. Can you underst the cat of reps, fix χ, $\underbrace{\hat{\chi}}_{\text{Jordan blocks}}$?

gen. inf.tesil char.

3. Vogan duality:

G $\qquad\qquad\qquad\qquad$ $\check{G}$

Get a reps here. $\qquad$ Langl'd par.

<u>Vogan</u>: Rep(real form, blocks) $\xleftrightarrow[\text{Koszul}]{}$ Rep(real form block)

character duality.

$\Gamma = \mathbb{Z}/2\mathbb{Z}$.

What should <u>Langlands parameter</u> be?

Categorically, this is { Koszul duality.

$\rightsquigarrow$ elementary

Flag mfd & orbits

Adams-Jullonx paper. (unreadable!)

$W_{\mathbb{R}} \to {}^L G$.
(Weil gp)

ABV book.

(p-adic version) After Vogan

Langlands parameters $\leftrightarrow$ perverse sheaves.

Fargues Conj: analogue $\mathbb{R}$.