2021年11月18日 14:39

Lie's Thm(s)

Section 0. Intro. / Overview:

Recall / Clearify some concepts:

Def. Lie Group: Smooth manifold G with smooth map $G\times G \to G$ (makes it a group)

Say it's { connected: connected as a manifold $\Rightarrow$ path connected
↳ which is locally path connected

simply connected: "Shrunk" of loop / continously deform paths / $\pi_1(G)=0$ ...

Def. For two paths $\gamma_1(t), \gamma_2(t)$ $t\in[0,1]$ in a connected component X of G

s.t. $\gamma_1(0)=\gamma_2(0)=A$ $\gamma_2(1)=\gamma_2(1)=B$

There $\exists$ continous function: $d:[0,1]\times[0,1]\to X$

① $d(0,t)=\gamma_1(t)$ ② $d(1,t)=\gamma_2(t)$ ③ $d(s,0)=A$ $d(s,1)=B$ $\forall s\in[0,1]$

Rmk. Some require simply connected to be (path) connected

If not, define on connected component and use "connected & simply connected"

Def. Lie algebra: vector space $\mathfrak{g}$ with a skew-bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (called Lie bracket)

satisfying Jacobi identity: $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$ $a,b,c\in\mathfrak{g}$

Ex. Lie bracket given by commutator $[a,b]=ab-ba$
↳ If composition is clear

Thm. (Lie's Thm, modern language)

$$\boxed{\text{Category of connected \& simply connected Lie groups}} \ni G \xrightarrow{\text{Functor } F} T_eG \in \boxed{\text{category of Lie algebras}}$$

is an equiv. of cats.

① What does F mean?

② Why does it make sense?

③ What should we do to get equivalence?

① $\begin{cases} \text{objects: } & G \overset{F}{\rightsquigarrow} T_eG \\ \text{morphisms: } & \end{cases}$

$$\begin{array}{ccc} G_1 & & T_eG_1 \\ \text{Lie group homo.} \downarrow \phi & \overset{F}{\rightsquigarrow} \theta & \downarrow (\text{Lie algebra homo.}) \\ G_2 & & T_eG_2 \end{array}$$

See ②

"taking derivative of smooth group homomorphism"

② 1) T_eG is a Lie algebra

2) Given Lie group homo. $\phi: G_1 \to G_2$

$\exists$ Lie alg. homo $\theta: T_eG_1 \to T_eG_2$ st $F(\phi) = \theta$ → "$d\phi$"

Covered in Section 1.

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

surj. "full"

inj. "faithful"

③ 1) Fully faithful: (First Lie Thm): Section 2

Given Lie alg. homo. $\theta: T_eG_1 \to T_eG_2$

$\exists$ $\boxed{\text{unique}}$ Lie group homo. $\phi: G_1 \to G_2$ s.t. $F(\phi) = \theta$

- Uniqueness (Determined locally!): exp & 1-para. subgroup
- Existence
  - Locally: CBH Thm(s)
  - Local to global:

Step 1. construction of $f(g)$ using path

Step 2. well-defined by path using Maurer-Cartan equation

Step 3. group homo. by concatenation.

2) Essentially surjective (Cartan-Lie Thm): Hard! Section 3.

$\forall$ (Finite dim) Lie alg. comes from a (connected & simply-connected) Lie group.

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Section 1.

① T_eG is a Lie algebra. "Left (/Right) invariant vector field"

Actually, T_eG is identical with "the Lie algebra of the Lie group"

Recall & Alternative view of vector field:

Def. A (smooth) vector field X on mfd M is a continuous (smooth) map $X: M \to TM = \bigsqcup_{p\in M} T_pM$ s.t. for projection $\pi: TM \to M$, $\pi \circ X = Id_M$ (section of π)

Rmk: $X_p \in T_pM$.

T_pM acts on $f \in C^\infty(M)$ at point p by derivation at p $C^\infty(M) \to \mathbb{R}$

X acts on $f \in C^\infty(M)$ by $Xf(p) = X_p f$

$$v(fg) = f(p)vg + g(p)vf.$$

Prop. X smooth $\Leftrightarrow$ Xf smooth for $f \in C^\infty(M)$

Lemma: For smooth coordinate chart (U, X^i) on M, $X: M \to TM$ smooth on U iff the component functions smooth

Pf (sketch):

"$\Rightarrow$" $Xf(x) = \left(X^i(x)\frac{\partial}{\partial x^i}\Big|_x\right)f$ smooth on $U \ni p$

"$\Leftarrow$" View x^i as smooth function on U

$\underset{\text{smooth}}{Xx^i} = X^j \frac{\partial}{\partial x^j}(x^i) = X^i$ component smooth

Def. A map $D: C^\infty(M) \to C^\infty(M)$ is called a derivation (of smooth function) if

Rmk: composable! smooth function on mfd. M

1) linear over $\mathbb{R}$
2) for $\forall f, g \in C^\infty(M)$ $D(fg) = fDg + gDf$ (Leibniz rule)

Prop. A map $D: C^\infty(M) \to C^\infty(M)$ is a derivation iff $Df = Xf$ for some smooth vector field X on M

pf. "$X \Rightarrow D$" define $f \mapsto Xf$ $C^\infty(M) \to C^\infty(M)$

at each point p $X_p(fg) = f(p)X_pg + g(p)X_pf$ tangent vector

"$D \Rightarrow X$" define vector field X by $X_pf = D\tilde{f}(p)$ RX_p: C M →

From now on, view vector field as derivation of $C^\infty(M)$

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Composition $XYf = X(Yf)$. Lie algebra structure!

$\to$ smooth vector field $\to$ derivation $C^\infty(M) \to C^\infty(M)$

Def. Lie bracket of $X, Y \in \mathfrak{X}(M)$: $[X,Y]f = XYf - YXf$

Prop. For $\forall X, Y \in \mathfrak{X}(M)$, we have $[X,Y] \in \mathfrak{X}(M)$

Pf. $\forall f, g \in C^\infty(M)$

$$
\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= \begin{array}{l} fXYg + gXYf - fYXg - gYXf \\ + YgXf + YfXg - XgYf - XfYg \end{array} \\
&= fXYg + gXYf - fYXg - gYXf \\
&= f[X,Y]g + g[X,Y]f
\end{aligned}
$$

Leibniz rule satisfied.

Rmk. Vector fields (derivation) have Lie algebra structure via Lie bracket

Why left-invariant vector fields are "the Lie algebra"? i.e. closed under Lie bracket

Def. For smooth map $\phi: M \to N$ mfd.

$X \in \mathfrak{X}(M)$ $Y \in \mathfrak{X}(N)$ are ϕ-related if $d\phi_p(X_p) = Y_{\phi(p)}$

"Left invariant" $\Leftrightarrow$ "L_g-related to itself"

Def. $X \in \mathfrak{X}(G)$ left invariant if $\forall g, g' \in G$

$d(L_g)_{g'}(X_{g'}) = X_{gg'}$ where $L_g(h) = gh$ Denote as $(L_g)_* X = X$

Lemma 1. X, Y ϕ-related iff $\forall f \in C^\infty(U)$ U open $\subseteq N$

$$X(f \circ \phi) = (Yf) \circ \phi$$

pf. at neighbourhood of $\phi(p)$

$$X(f\circ\phi)(p) = X_p(f\circ\phi) = d\phi_p(X_p)f$$

$$\left(\left(Yf \circ \phi(p) = Yf(\phi(p)) = Y_{\phi(p)} f \right)\right)$$

Lemma 2.

If $\mathfrak{X}(M) \ni \begin{array}{c} X_1 \dashrightarrow Y_1 \\ \phi\text{-related} \\ X_2 \dashrightarrow Y_2 \end{array} \in \mathfrak{X}(N)$ $\quad \phi: M \to N$

Then $\mathfrak{X}(M) \ni [X_1, X_2]$ ϕ-related to $[Y_1, Y_2] \in \mathfrak{X}(N)$

$$X_1X_2(f\circ\phi) = X_1(Y_2 f)\circ\phi) = (Y_1Y_2 f)\circ\phi \quad \text{(Lemma 1)}$$
$$X_2X_1(f\circ\phi) = \cdots = (Y_2Y_1 f)\circ\phi$$

(Lemma 1)

$$\Rightarrow [X_1,X_2](f\circ\phi) = X_1X_2(f\circ\phi) - X_2X_1(f\circ\phi) = (Y_1Y_2f)\circ\phi - (Y_2Y_1f)\circ\phi = ([Y_1,Y_2]f)\circ\phi$$

Coro. For $X_1, X_2 \in \mathfrak{X}(G)$ left invariant, $[X_1,X_2]$ is also left invariant

i.e. Left-invariant vector field is Lie algebra. Denoted $Lie(G)$

Prop. T_eG identical with $Lie(G)$ i.e. T_eG is a Lie algebra.

Pf. evaluation map ε : $Lie\, G \to T_eG$, $\varepsilon(X) = X_e$

inj. $\varepsilon(X) = X_e = 0 \Rightarrow X_g = d(L_g)_e(X_e) = 0 \ \forall g\in G$

surj. for $v\in T_eG$, define by $X^v_g := d(L_g)_e v$ a vector field X^v

Check: 1) smooth $(X^v f)(g) = X^v_g f = d(L_g)_e(v) f = v(f\circ L_g)$

$\gamma(0)=1 \quad \gamma'(0)=v$ $\qquad = \gamma'(0)(f\circ L_g) = \frac{d}{dt}\Big|_{t=0}(f\circ L_g\circ\gamma)(t)$

smooth

2) left-invariant

i.e. $d(L_h)_g(X^v_g) = X^v_{hg} = d(L_{hg})_e(v)$

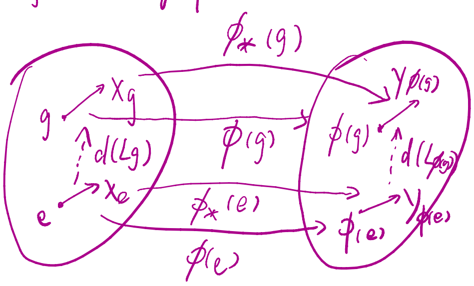
$$\| \qquad\qquad\qquad \|$$

$$(dL_h)_g\circ d(L_g)_e(v) = d(L_h\circ L_g)_e(v)$$

② Induce Lie algebra homo. from Lie group homo.

$$\begin{tikzcd} G_1 \arrow[r, "\phi"] \arrow[d, dashed] & G_2 \\ Lie(G_1) \arrow[r, "\theta"] & Lie(G_2) \end{tikzcd}$$

Left inv. vector field



Prop. For $\forall\ X\in Lie(G_1)$, $\exists$ unique $Y\in Lie(G_2)$ which is ϕ-related to X. Denoted $\phi_* X$.

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Pf. let $Y = d\phi_e(X_e)^V$ i.e. $Y_g = d(L_g)_e(d\phi_e(X_e))$

If any $Y \in Lie(G)$ is ϕ-related to X it must satisfy $Y_e = d\phi_e(X_e)$ hence uniquely determined in this form.

ϕ related to X:

$\phi(gg') = \phi(g)\phi(g')$ group homo

$\Rightarrow \phi(L_g g') = L_{\phi(g)}\phi(g')\quad \forall g'$

$\Rightarrow \phi \circ L_g = L_{\phi(g)} \circ \phi$ $\quad d(\phi\circ L_g)_{g'}(v)(f) = v(f\circ\phi\circ L_g) = (dL_g)_{g'}(v)(f\circ\phi)$

$= (d\phi)_{L_g g'}(dL_g)_{g'}(v)f$

$\Rightarrow d\phi \circ d(\underset{\uparrow X_e}{L_g}) = d(L_{\phi(g)}) \circ d\underset{\uparrow X_e}{\phi}$

$\Rightarrow d\phi(X_g) = Y_{\phi(g)}$

Coro. $\phi_*[\underbrace{X_1, X_2}_{\in Lie(G_1)}] = [\underbrace{\overset{Y_1}{\phi_* X_1}, \overset{Y_2}{\phi_* X_2}}_{\in Lie(G_2)}]$ by Lemma 2.

i.e. ϕ_* is Lie algebra homo. (induced by Lie group homo. ϕ)

$\phi_* = d\phi$

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Section 2.

Thm. (First Lie's Thm)

For connected & simply connected Lie groups G_1, G_2

Given Lie alg. homo. $\theta: T_eG_1 \to T_eG_2$

$\exists$ unique Lie group. homo. $\phi: G_1 \to G_2$ s.t. $F(\phi) = \theta$

Uniqueness

Claim: Given a Lie algebra homo. $\theta: T_eG_1 \to T_eG_2$ $d\phi = \theta$

There's at most one Lie group homo. $\phi: G_1 \to G_2$ inducing it

Pf. A Lie group homo. is determined by its restriction to neighborhood of e. e.g. $\exp\xi$ where $\xi \in T_eG_1$

$t \rightsquigarrow \phi(\exp t\xi)$ is one-para. subgroup of G_2 with derivative at $t=0$: $d\phi_e(\xi)$

$\parallel$

$t \rightsquigarrow \exp t\theta(\xi)$ is one-para subgroup of G_2 with derivative at $t=0$: $\theta(\xi)$

$\Rightarrow$ $\phi(\exp\xi) = \exp\theta(\xi)$ uniquely determined by θ

Existence

$$\begin{tikzcd}
G_1 \arrow[r, "\phi"] & G_2 \\
T_eG_1 \arrow[u, shift left, "\exp"] \arrow[r, "\theta"] & T_eG_2 \arrow[u, "\exp"']
\end{tikzcd}$$

$\xrightarrow{}$ group homo?

$\log$ locally

Locally, it's group homo. by C-B-H Thms

Prop. $\exists$ neighbourhood $U \ni e$ in T_eG, s.t. $\forall v \in U$

$\phi: G_1 \to G_2$, /

Prop. $\exists$ neighbourhood $U \ni e$ in $\mathrm{Lie}\,G$, s.t. $\forall v \in U$

$\phi : G_1 \to G_2$, $\phi(\exp v) = \exp \theta(v)$ is a group homo.

$$
\begin{array}{ccc}
\phi(\exp u \cdot \exp v) & = & \phi(\exp u)\cdot \phi(\exp v) \\
\| \leftarrow \text{(Geometric) CBH} & & \| \\
\phi(\exp(u \circ v)) & = & \exp\theta(u)\cdot \exp\theta(v) \\
\| \leftarrow \text{Def} & & \| \\
\exp \theta(u \circ v) & = & \exp(\theta(u) \circ \theta(v)) \\
 & \uparrow &
\end{array}
$$

θ Lie alg homo.

& converges absolutely of power series representation

Notations: $\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{K}\langle\langle X, Y\rangle\rangle$ ring of formal power series with non-commutative variables X, Y

Formal power series: (in $\mathbb{K}\langle\langle X\rangle\rangle$)

$$\exp(X) := 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots$$

$$\log(1+X) := 1 - \frac{X^2}{2} + \frac{X^3}{3} \mp \cdots + (-1)^n \frac{X^n}{n} \mp \cdots$$

Lemma 1. (Algebraic CBH)

The product in logarithmic coordinates lies in the Lie sub-algebra of $K\langle\langle X,Y\rangle\rangle$

Explicitly: $\log(\exp X \cdot \exp Y) = X + Y + \cdots + \frac{1}{2}[X,Y] + \frac{1}{12}[[X,Y],Y] + \frac{1}{12}[[Y,X],X] + \cdots$

Denote the homogenous-n term $h_n(X,Y)$

Lie bracket by commutator

Lemma 2 (Geometric CBH):

For Lie group G $\exp : T_eG \to G$

$\exists$ neighbourhood U of origin in T_eG s.t. $\forall u, v \in U$

$u \circ v = \sum_{n=1}^{\infty} h_n(u,v)$ converges absolutely, and $\exp(u \circ v) = \exp(u)\exp(v)$

i.e. lie in the Lie subalgebra

To global : $\left\{ \begin{array}{l} \text{construction} \\ \text{well-defined} \\ \text{group homo.} \end{array} \right.$

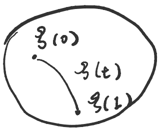
Step 1. Construction

Path connected : For $\forall g \in G_1$ $\exists$ path

$\gamma : [0,1] \to G_1$ s.t. $\gamma(0) = e$ $\gamma(1) = g$

Want to define $\phi(g)$ in G_2

In Lie alg. T_eG_1 :



have a path

$$t \rightsquigarrow \gamma'(t)\gamma(t)^{-1} = \xi(t) \in T_eG_1$$

$$\Big\downarrow \theta \quad \text{Lie alg. homo.}$$

$$t \rightsquigarrow \theta(\xi(t)) = \bar{\xi}(t) \in T_eG_2$$

Define a path in G_2 : $\varphi : [0,1] \to G_2$ s.t.

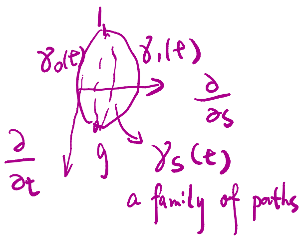
$$\varphi'(t) = \bar{\xi}(t)\,\varphi(t) \quad \& \quad \varphi(0) = e$$

Then $\varphi(1) = \phi(g)$

Doesn't depend on the choice of path in G_1 (well-defined)

Simply connected Def. $\xi(t,s) = \frac{\partial}{\partial t}\gamma_s(t)\,\gamma_s(t)^{-1} \in T_eG_1$

$$\eta(t,s) = \frac{\partial}{\partial s}\gamma_s(t)\,\gamma_s(t)^{-1} \in T_eG_1$$



Want : $\frac{\partial \varphi}{\partial s} = 0$ when $t = 1$ i.e. $\varphi(1,s)$ independent of s

Step 2.

Lemma (Maurer - Cartan Equation)

$$\frac{\partial \xi}{\partial s} - \frac{\partial \eta}{\partial t} = [\eta, \xi]$$

Pf. (skip)

$\xi : [0,1] \times [0,1] \to T_eG$

$\xi : [0,1]\times[0,1] \to T_eG$

Let $\alpha(i) = \gamma_s(i)\gamma_s(t)^{-1}$ $\quad \alpha(t) = e$

$$\xi(t,s)(f) = \left(\frac{\partial}{\partial i} f\circ \gamma_s(i)\,\gamma_s(t)^{-1}\right)(t) = (f\circ\alpha)'(s)$$

$$\frac{\partial \xi}{\partial s}(f) = \frac{\partial}{\partial s}\left(\left(\frac{\partial}{\partial i} f\circ\gamma_s(i)\,\gamma_s(t)^{-1}\right)(t)\right)$$

$\eta : [0,1]\times[0,1] \to T_eG$

Let $\alpha(h) = \gamma_h(t)\gamma_s(t)^{-1}$ $\quad \alpha(s) = e$

$$\eta(t,s)(f) = \left(\frac{\partial}{\partial h} f\circ\gamma_h(t)\,\gamma_s(t)^{-1}\right)(s)$$

$$\frac{\partial \eta}{\partial t}(f) = \frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial h} f\circ \gamma_h(t)\,\gamma_s(t)^{-1}\right)(s)\right)$$

$[\eta,\xi]$ kind of "abusing notation"

really mean: $[\eta,\xi]f = \eta X^{\xi} f - \xi X^{\eta} f$

$$X^{\xi}_g = d(R_g)_e\,\xi \in T_gG \xrightarrow{d(R_{g^{-1}})} T_eG$$

$$(X^{\xi} f(g))(g') = \xi f(g' g^{-1})$$

$$X^{\xi} f = \left(\frac{\partial}{\partial i} f\circ\gamma_s(i)\,\gamma_s(t)^{-1} g^{-1}\right)(t)$$

when $g = \alpha(h) = \gamma_h(t)\gamma_s(t)^{-1}$

$$\eta X^{\xi} f = \frac{\partial}{\partial h}\left(\frac{\partial}{\partial i} f\circ\gamma_s(i)\,\gamma_s(t)^{-1}\circ(\gamma_h(t)\gamma_s(t)^{-1})^{-1}\right)$$

$$= \frac{\partial}{\partial h}\left(\frac{\partial}{\partial i} f\circ\gamma_s(i)\,\gamma_h^{-1}(t)\right)$$

$$\xi X^{\eta} f = \frac{\partial}{\partial i}\left(\frac{\partial}{\partial h} f\circ\gamma_h(t)\,\gamma_s(i)^{-1}\right)$$

Prop. $\dfrac{\partial\bar\xi}{\partial s} - \dfrac{\partial\bar\eta}{\partial t} = [\bar\eta,\bar\xi]$

$\Leftrightarrow$ For $\varphi : [0,1]\times[0,1]\to G$

the differential equations $\dfrac{\partial\varphi}{\partial t} = \bar\xi\varphi$ $\dfrac{\partial\varphi}{\partial s} = \bar\eta\varphi$ have solution φ.

[illegible] $\varphi(0,s) = e$ & $\varphi(1,s) = g$)

the differential equations $\frac{\partial \varphi}{\partial t} = \xi \varphi$ $\frac{\partial \varphi}{\partial s} = \eta \varphi$ have solution φ.

(with initial conditions $\varphi(0,s) = e$ & $\varphi(1,s) = g$)

Pf. (skip) Formal verification "abusing notation"

"$\Leftarrow$" $\frac{\partial \varphi}{\partial t} = \bar{\xi}\varphi$ $\frac{\partial \varphi}{\partial s} = \bar{\eta}\varphi$

$$\frac{\partial}{\partial s}\left(\frac{\partial \varphi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial s}\right) \Rightarrow \frac{\partial}{\partial s}(\bar{\xi}\varphi) = \frac{\partial}{\partial t}(\bar{\eta}\varphi)$$

$$\Rightarrow \frac{\partial \bar{\xi}}{\partial s}\varphi + \bar{\xi}\frac{\partial \varphi}{\partial s} = \frac{\partial \bar{\eta}}{\partial t}\varphi + \bar{\eta}\frac{\partial \varphi}{\partial t}$$

$$\Rightarrow \left(\frac{\partial \bar{\xi}}{\partial s} - \frac{\partial \bar{\eta}}{\partial t}\right)\varphi = \bar{\eta}\frac{\partial \varphi}{\partial t} - \bar{\xi}\frac{\partial \varphi}{\partial s} = [\bar{\eta}, \bar{\xi}]\varphi$$

"$\Rightarrow$" $\varphi(t,0) = \int_{s=0} \bar{\xi}\varphi\, dt$ $\varphi(t,s) = \int_{t=c} \bar{\eta}\varphi\, ds$

$$\frac{\partial}{\partial s}\left(\frac{\partial \varphi}{\partial t} - \bar{\xi}\varphi\right) = \frac{\partial \varphi}{\partial t \partial s} - \frac{\partial \bar{\xi}}{\partial s}\varphi - \bar{\xi}\frac{\partial \varphi}{\partial s}$$

$$= \frac{\partial}{\partial t}(\bar{\eta}\varphi) - \frac{\partial \bar{\xi}}{\partial s}\varphi - \bar{\xi}\frac{\partial \varphi}{\partial s}$$

$$= \left(\frac{\partial \bar{\eta}}{\partial t} - \frac{\partial \bar{\xi}}{\partial s}\right)\varphi + \underbrace{\left(\bar{\eta}\frac{\partial}{\partial t} - \bar{\xi}\frac{\partial}{\partial s}\right)\varphi}_{= [\bar{\eta}, \bar{\xi}]\varphi}$$

Note that when $t=1$, η and $\bar{\eta}$ vanish by def.

Coro. when $t=1$ $\frac{\partial \varphi}{\partial s} = 0$. $\Rightarrow$ $\varphi(1,s)$ independent of s.

Step 3. group homo.

For $g, g' \in G_1$ paths $\begin{matrix} \gamma_\xi(t) \\ \gamma_\eta(t) \end{matrix}$ $[0,1] \to G_1$ $\begin{matrix} \gamma_\xi(0) = e & \gamma_\xi(1) = g \\ \gamma_\eta(0) = e & \gamma_\eta(1) = g \end{matrix}$

$\xi(t) = \dot{\gamma}_\xi(t)\gamma_\xi(t)^{-1} \in T_eG_1$ $\eta(t) = \dot{\gamma}_\eta(t)\gamma_\eta(t)^{-1} \in T_eG_1$

Def. Concatenation of ξ and η is

$$(\eta * \xi)(t) = \begin{cases} 2\xi(2t) & 0 \le t \le \frac{1}{2} \\ 2\eta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

As in step 1, $\begin{matrix} \varphi_\xi'(t) = \xi(t)\varphi_\xi(t) & \varphi_\xi(1) = g \\ \varphi_\eta'(t) = \eta(t)\varphi_\eta(t) & \varphi_\eta(1) = g' \end{matrix}$

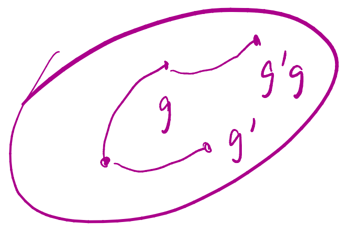
$\varphi_\eta'(t) = \eta(t)\varphi_\eta(t) \qquad \varphi_\eta(1) = g'$

Let $\varphi_{\eta*\xi}'(t) = (\eta*\xi)(t)\,\varphi_{\eta*\xi}(t) \qquad \varphi_{\eta*\xi}(0) = e \Rightarrow \varphi_{\eta*\xi}(1) = \varphi_\eta(1)\cdot\varphi_\xi(1)$

Actually solving it gives $\varphi_{\eta*\xi}(t) = \begin{cases} \varphi_\xi(2t) & 0 \le t \le \frac{1}{2} \\ \varphi_\eta(2t-1)\varphi_\xi(1) & \frac{1}{2} \le t \le 1 \end{cases}$

Prop. The ϕ constructed by φ in step 1 is group homo.

pf. $\phi(g')\phi(g) = \varphi_\eta(1)\varphi_\xi(1) = \varphi_{\eta*\xi}(1) = \phi(g'g)$



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Section 3.

For $\forall$ (finite dim) Lie algebra $\mathfrak{g}$

Find a Lie group G s.t. $\mathrm{Lie}(G) = \mathfrak{g}$

Remark: Here we are talking about Arbitrary, Abstract Lie algebras

i.e. Vector space + skew bilinear form + Jacobi identity

We do not yet know it's in TeG and so that we can use exponential map

(Well, this theorem is exactly telling us this is however true indeed...)

With the help of Ado's Thm:

$\forall$ (finite dim) Lie algebra is iso. to a subalgebra of the Lie alg. $M_n\mathbb{R}$.

Def. $\mathcal{P} = \left\{ \begin{array}{l} \xi : [0,1] \to \mathfrak{g} \subseteq M_n\mathbb{R} \\ \xi(0) = \xi(1) = 0 \\ \xi'(0) = \xi'(1) = 0 \end{array} \right\}$

Def. For some ξ above, $\varphi_\xi : [0,1] \to GL_n\mathbb{R}$ is the solution of $\varphi_\xi'(t) = \xi(t)\varphi_\xi(t)$ with $\varphi_\xi(0) = 1$

Prop. $\mathcal{P}/\sim$ where $\xi_0 \sim \xi_1 \iff \varphi_{\xi_0}(1) = \varphi_{\xi_1}(1)$ is a Lie group with group structure by concatenation

Def. For $\xi \in U \subseteq \mathcal{P}$ where U is a neighbourhood of Id

Let $\widehat{\varphi_\xi} = \log \varphi_\xi \in M_n\mathbb{R}$ For some $\varphi_\xi \in \{\varphi_\xi\}_{\xi \in \mathcal{P}}$

Prop. $\mathcal{P}/\sim$ is locally homeomorphic to $\mathfrak{g}$

Claim: If $\widehat{\varphi_\xi} \in \mathfrak{g}$, then above prop. holds

$\mathcal{P} \supseteq U \ni \xi \overset{1:1}{\longleftrightarrow} \varphi_\xi \in V \subseteq \{\varphi_\xi\}_{\xi\in\mathcal{P}}$

$\xi_0 \sim \xi_1 \iff \varphi_{\xi_0}(1) = \varphi_{\xi_1}(1)$

$U \cong W$ $\quad\Leftarrow\quad$ $\log \downarrow$ locally iso. $V \cong W$

$W \supseteq \{\widehat{\varphi_\xi}\}_{\xi\in\mathcal{P}} \qquad \{\widehat{\varphi_\xi}\}_{\xi\in\mathcal{P}}$

$\widehat{\varphi_0} \sim \widehat{\varphi_1} \overset{2}{\Longrightarrow} \widehat{\varphi_0}(1) = \widehat{\varphi_1}(1)$

Lemmas: For $\forall$ function $A : [0,1] \to M_n\mathbb{R}$ (Skip)

$$\frac{d}{dt}(e^A) = \int_0^1 e^{sA}\left(\frac{dA}{dt}\right)e^{(1-s)A}\,ds$$

$$e^{\operatorname{ad}A}(B) := e^A B e^{-A}$$

$$\frac{d}{dt}(e^A)e^{-A} = \sum_{k\geq 0}\frac{\operatorname{ad}A}{(k+1)!}\frac{d}{dt}A$$

Prop. $\widehat{\varphi_{\xi(t)}}$ is a solution to differential equation

$$\sum_{k\geq 0}\frac{\operatorname{ad}A(t)}{(k+1)!}A'(t) = \xi(t) \qquad \text{plug in } \uparrow$$

But we already have $\varphi'_\xi(t)\varphi_\xi(t)^{-1}$ as solution $\uparrow$

$\therefore \quad \widehat{\varphi_\xi}(t) = \varphi'_\xi(t)\varphi_\xi(t)^{-1} \in \mathfrak{g} \qquad \square$

Prop. The Lie algebra of $\mathcal{P}/\sim$ is $\mathfrak{g}$

Prop. The Lie algebra of $\mathcal{P}/\sim$ is $\mathfrak{g}$

Pf. $G = \mathcal{P}/\sim \xrightarrow{\varphi} GL_n\mathbb{R}$ is a Lie group homo.

$\xi \rightsquigarrow \varphi_\xi(1)$ by concatenation

(Induce Lie alg. homo.)

$$T_eG \xrightarrow{d\varphi} M_n\mathbb{R}$$

$$d\varphi\,[\xi_1, \xi_2]_{T_eG} = [d\varphi\,\xi_1, d\varphi\,\xi_2]_{M_n\mathbb{R}}$$

lies in $\mathfrak{g}$

$$\|$$

$$\mathfrak{g} \xrightarrow{\iota} M_n\mathbb{R}$$

$$\iota[\xi_1, \xi_2]_{\mathfrak{g}} = [\iota\,\xi_1, \iota\,\xi_2]_{M_n\mathbb{R}}$$