The University of Melbourne
Representation Theory Seminar

LIE THEORY

Smooth Manifolds, Tangent space, One Parameter Subgroups and the Exponential Map

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1 Preamble

So far in this seminar we have come across the notion of a Lie group[1] many times whilst discussing other topics. Each time we gave only a heuristic or informal definition of what it means to be a lie group before proceeding on with the talk. The aim of these three lectures will be to construct the objects central to Lie Theory whilst hopefully giving insight into how the relations between them. It is important to note that throughout this discussion we will be assuming basic results and definitions from an undergraduate course in topology and group theory. With that out of the way we begin by introducing manifolds, the objects core to differential geometry.

2 Manifolds

2.1 Topological Manifolds

Definition 2.1. *Let M be a topological space, we say that M is a **topological manifold** of dimension n if*

- *M is Hausdorff: For every pair of distinct points $p, q \in M$ there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*
- *M is second-countable: There exists a countable basis for the topology of M.*
- *M is locally Euclidean of Dimension n: Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Example 2.1. *The empty set $\emptyset$ is a topological manifold.*

Example 2.2. *The vector space $\mathbb{R}^n$ with metric topology is a topological manifold.*

Definition 2.2. *Let M be a topological n-manifold. A **coordinate chart** on M is a pair (U, φ) where*

- *The **coordinate neighbourhood** $U \subset M$ is an open subset.*
- *The **(local) coordinate map** $\varphi : U \subset M \to \hat{U} \subset \mathbb{R}^n$ is a homeomorphism.*

*For a point $p \in U$, the coordinate map $\varphi(p) = (x^1(p), x^2(p), ..., x^n(p))$ has component functions $(x^1, x^2, ..., x^n)$ which are called the **local coordinates** on U.*

[1]'Lie' is pronounced 'Lee'

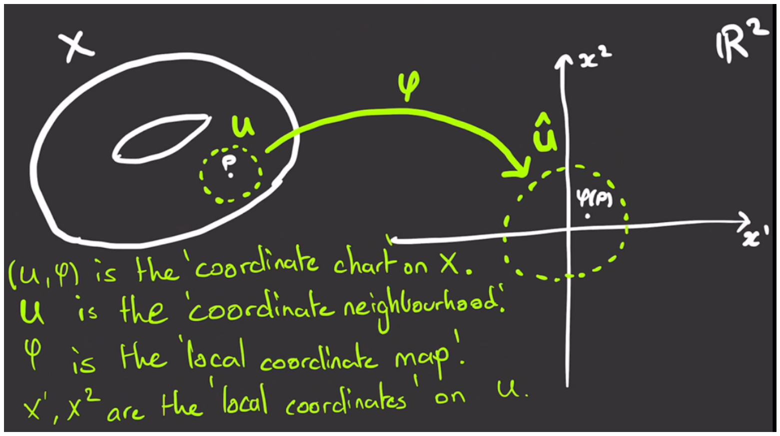


Figure 1: A coordinate chart of a topological manifold X

Definition 2.3. *Let M be a topological n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$ then the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the* ***transition map*** *from φ to ψ.*

The transition map is a composition of homeomorphisms, and is therefore itself a homeomorphism.

Example 2.3. *Let the punctured plane $\mathbb{R}^2 \setminus \{O\}$ be our topological manifold. We have a chart defined globally on this manifold given by*

$$\varphi : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \text{ where } P \mapsto (x, y) \text{ such that } (x, y) \neq (0, 0)$$

Now if we let U be the (open) upper right quadrant of our punctured plane containing the point P.We have the chart

$$\psi : U \to \mathbb{R}^2 \text{ where } P \mapsto (r, \theta) \text{ such that } r > 0 \text{ and } \theta \in (0, \frac{\pi}{2})$$

Then the transition maps defined on the overlap $U \cap \mathbb{R}^2 \setminus \{O\}$ are the familiar results for the change of coordinates between Cartesian and polar coordinates.

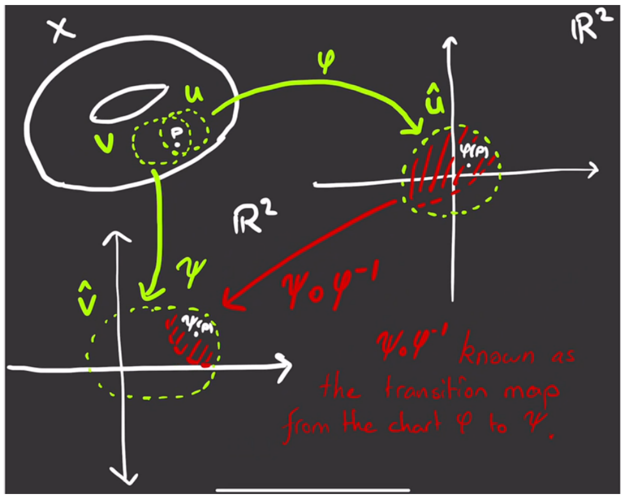
$$(\varphi \circ \psi^{-1})(r, \theta) = (rsin(\theta), r\cos(\theta))$$

$$(\psi \circ \varphi^{-1})(x, y) = (\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right))$$

Hopefully the above example makes it clear that we in developing this theory we have taken the conceptual leap of decoupling the coordinates with the points of the underlying manifold.

2.2 Smooth Manifolds

Definition 2.4. *Two charts* (U, φ) *and* (V, ψ) *are said to be* ***smoothly compatible*** *if either* $U \cap V = \emptyset$ *or the transition map* $\psi \circ \varphi^{-1}$ *is a diffeomorphism.*



Definition 2.5. *We define an* ***atlas for M*** *to be a collection of charts* $\{(U_i, \varphi_i)\}_{i \in I}$ *of which the coordinate neighbourhoods form a cover of* M

Definition 2.6. *An atlas* $\mathcal{A}$ *is called a* ***smooth atlas*** *if any two charts in* $\mathcal{A}$ *are smoothly compatible.*

Definition 2.7. *A smooth atlas* $\mathcal{A}$ *on* M *is said to be a* ***maximal smooth atlas*** *if it is not properly contained in any larger smooth atlas.*

Definition 2.8. *A* ***smooth structure*** *on a topological manifold is a maximal smooth atlas* $\mathcal{A}$*.*

Definition 2.9. *A* ***smooth manifold*** *is a pair* $(M, \mathcal{A})$ *where* M *is a topological manifold and* $\mathcal{A}$ *is a smooth structure on* M*.*

Not all topological manifolds necessary admit smooth structures. [2]

[2] The first example of a topological manifold that admits no smooth structure was discovered in 1960 by Michel Kervaire.

2.3 Examples of Smooth Manifolds

Example 2.4. *Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a smooth function. The graph of a smooth function $(\Gamma(f), \mathcal{A})$ is a smooth manifold. The graph of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by*

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\}$$

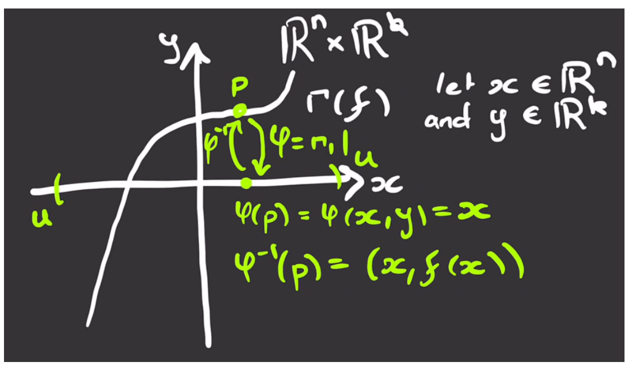
with the subspace topology. Since it is a subset of $\mathbb{R}^{n+k}$ is satisfies the conditions of being Hausdorff and second countable. Consider the projection map $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, it is not injective and so it does not admit an inverse so let $\varphi : \Gamma(f) \to U$ where $\varphi = \pi_1|_{\Gamma(f)}$

$$\varphi(x, y) = x \quad (x, y) \in \Gamma(f)$$

Because φ is the restriction of a continuous map, it is continuous and it is a homeomorphism because it has continuous inverse given by

$$\varphi^{-1}(x) = (x, f(x))$$

$(\Gamma(f), \varphi)$ is a global coordinate chart called graph coordinates. The chart is smooth and so we have the smooth structure $\mathcal{A} = \{\varphi\}$ Thus the graph of a smooth function is a smooth manifold $(\Gamma(f), \mathcal{A})$ of dimension n.



Example 2.5. *The n-sphere S^n is a smooth manifold.* ***n-Sphere****. The unit sphere S^n is hausdorff and second countable because it is a topological subspace of $\mathbb{R}^{n+1}$. For each index $i = 1, .., n+1$ let U_i^+ and U_i^- denote the subsets of $\mathbb{R}^{n+1}$ where the i-th coordinate is positive and negative respectively:*

$$U_i^+ = \{(x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\} \quad U_i^- = \{(x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} : x^i < 0\},$$

with charts respectively given by $\varphi_i^\pm : U_i^\pm \cap S^n \to \mathbb{B}^n$ where

$$\varphi_i^\pm(x^1, .., x^{n+1}) = (x^1, ..., x^{i-1}, x^{i+1}, ..x^{n+1}).$$

The continuous inverse maps are $(\varphi_i^\pm)^{-1} : \mathbb{B}^n \to U_i^\pm \cap S^n \subset \mathbb{R}^{n+1}$

$$(\varphi_i^\pm)^{-1}(x^1, .., x^{i-1}, x^{i+1}, .., x^{n+1}) = (x^1, .., x^{i-1}, \pm f(u), x^{i+1}, .., x^{n+1})$$

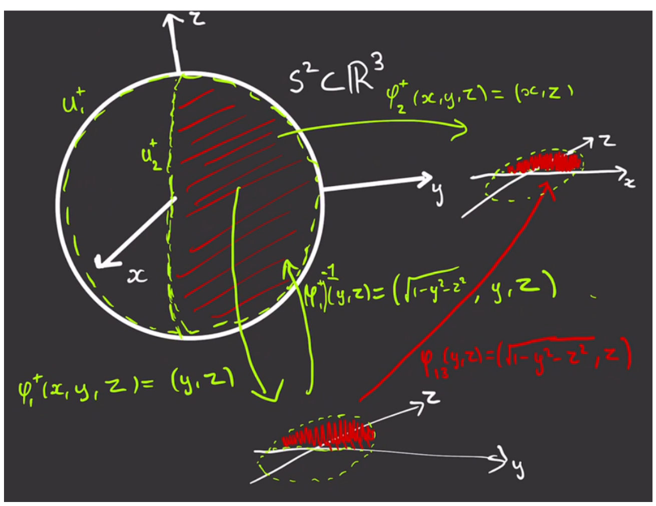
where $f : \mathbb{B}^n \to \mathbb{R}$ is the continuous function

$$f(u) = \sqrt{1 - |u|^2}$$

Thus S^n is a topological manifold. We get the transition maps by considering the charts φ_i and φ_j and then composing $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1} : \mathbb{B}^n \to \mathbb{B}^n$ we get

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1} = \begin{cases} (x^1, .., x^{i-1}, x^{i+1}, .., \pm f(u), .., x^n), & \text{if } i < j \\ Id_{\mathbb{B}^n}, & \text{if } i = j \\ (x^1, .., \pm f(u), .., x^{i-1}, x^{i+1}, .., x^n), & \text{if } i > j \end{cases}$$

all of which are smooth. Therefore the n-sphere is a smooth manifold.



Exercise 2.1. *Show that a finite dimensional real vector space V with a topology induced by the norm is a smooth manifold.*

Exercise 2.2. *Suppose $M_1, ..., M_k$ are topological manifolds of dimensions $n_1, ..., n_k$ respectively. Show the product space given by $M_1 \times M_2 \times ... \times M_k$ is a smooth manifold with dimension $n_1 + n_2 + ...n_k$.*

Example 2.6. *The space of real $m \times n$-matrices $M(m \times n, \mathbb{R})$ is a real vector space of dimension mn under matrix addition and scalar multiplication and therefore is a smooth manifold. (We can write any $m \times n$-matrix as a tuple of $m \times n$ entries so we can identify the space of matrices with $\mathbb{R}^{mn}$.)*

Example 2.7. *Let U be any open subset of $\mathbb{R}^n$ then U is a topological $n-$manifold with a single chart $\{U, Id_U\}$ which also defines a smooth structure on U.*

Example 2.8. *The General Linear Group $GL(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is an open sub-manifold of $M(m \times n, \mathbb{R})$ and therefore is a smooth manifold.*

3 Lie Groups

Definition 3.1. *A **Lie-Group** is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by*

$$m(g, h) = gh,\ i(g) = g^{-1},$$

are both smooth.[3]

Example 3.1. *The circle group $S^1 \subset \mathbb{C}^*$ parameterised by θ is a lie group where multiplication and inversion have the smooth coordinate expressions*

$$m(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1+\theta_2)}, \quad i(e^{i\theta}) = e^{-i\theta}$$

Example 3.2. *The general linear group $GL(n, \mathbb{R})$ with matrix multiplication and the inversion map given by cramers rule is a lie group.*

$$m(A, B) = A \cdot B \quad i(A) = \frac{\mathrm{adj}(A)}{\det(A)}$$

[3]Hilbert's fifth problem concerned whether being 'smooth' was a requirement in the definition. Both Von Neumann and Lev Potryagin made initial progress on the problem in the 1930's and it wasn't until 1953 when it was shown Gleason, Montgomery and Zippin that such a restriction is unnecessary. Furthermore the inversion map being smooth is can be derived from applying the implicit function theorem to $xy = 1$. However we have not mentioned the implicit function function theorem thus far and so we have not omitted the condition from our definition of Lie group.

Examples of Lie groups (such as $O(n), U(n), SO(n), SU(n), SL(\mathbb{R})$ etc.) were the subjects of three previous talks, so for more in depth examples review the notes of Benjamin Gerraty or Eskander Salloum.

Definition 3.2. *If G and H are Lie groups then a* ***Lie group homomorphism*** *from $F; G \to H$ is a smooth map that is also a group homomorphism.*

Definition 3.3. *If F is a diffeomorphism then F is a an* ***isomorphism of Lie groups*** *G and H.*

Example 3.3. *Consider the map $\exp : \mathbb{R} \to \mathbb{R}^*$ from the additive real group to the multiplicative real group given by $\exp(t) = e^t$. It is smooth and a Lie group homomorphism since*

$$e^{s+t} = e^s e^t \text{ where } s, t \in \mathbb{R}$$

Here the image of the exponential map is not surjective. However consider $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ then $\exp$ is a lie group diffeomorphism with inverse given by $\log : \mathbb{R}_{>0} \to \mathbb{R}$. So $\exp$ here is an isomorphism and $\mathbb{R} \simeq \mathbb{R}^*$.

Example 3.4. *Consider the operators left translation and right translation on a lie group G given by*

$$L_g : G \to G, \text{ where } L_g(h) = gh \text{ and } R_g : G \to G, \text{ where } R_g(h) = hg, \text{ with } g, h \in G$$

respectively called ***left translation*** *and* ***right translation****. These are smooth functions since multiplication is smooth and moreover isomorphisms as they are diffeomorphisms with inverse given by $L_{g^{-1}}$.*

4 Tangent Space

There are many different ways to define the tangent space all of which have varying degrees of abstraction, however they can all be shown to be equivalent.

Definition 4.1. *Let M be a smooth manifold a curve in M is a differentiable map $\gamma : J \to M$ where $J \subset \mathbb{R}$ is an interval containing $0 \in J$.*

Definition 4.2. *Suppose P is a point of M. Let γ_1 and γ_2 be two curves on our manifold such that $\gamma(0) = P$. We define $\gamma_1 \sim \gamma_2$ if*

$$\frac{d}{dt}\Big|_{t=0}(f(\gamma_1(t))) = \frac{d}{dt}\Big|_{t=0}(f(\gamma_2(t)))$$

*for every smooth real valued real function f defined in a neighbourhood of P. Denote one of these equivalence classes by $\gamma_1'(0)$ this is a **tangent vector** at P. The set of these equivalence classes defines the **tangent space** of M at P written T_pM.*

One can then show that the tangent space is a n-dimensional vector space over $\mathbb{R}$ with origin identified as the point P of M. The basis vectors of this space can be identified with the partial derivative operators giving us

$$T_PM = \mathrm{Span}_{\mathbb{R}}\{\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n}\}$$

Therefore vectors are operators that act on functions which have a domain containing a neighbourhood of P.

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_P \in T_PM$$

Definition 4.3. *If M and N are smooth manifolds and $F : M \to N$ is a smooth map for each $p \in M$ we define the **differential** of F at p (or the **pushforward** of F at p) to be the map $dF : T_pM \to T_pN$ defined by*

$$dF_p(\gamma'(0)) = (F \circ \gamma)'(0)$$

where γ is a curve on M such that $\gamma(0) = P$. The pushforward of the tangent vector $\gamma'(0)$ to the curve is the tangent vector of the curve $F \circ \gamma$ at 0.

Alternatively if we consider tangent vectors to be derivations acting on smooth real valued functions then the differential is given by

$$dF(X)(f) = X(f \circ F)$$

where $f \in C^\infty(N)$ and an arbitrary derivation $X \in T_pM$ at a point $p \in M$.

Note all tangent spaces on M have the same dimension and thus are isomorphic as vector spaces where the pushforward and its inverse (referred to as the **pullback**) maps give the isomorphisms. Recall if we let g and h be elements of the Lie group G, we can define the operators Left translation by g and right translation by g given by

$$L_g : G \to G,\ L_g(h) = gh \text{ and } R_g : G \to G,\ R_g(h) = hg$$

respectively. These maps are smooth and in fact they are diffeomorphisms since L_g has inverse $L_{g^{-1}}$ and R_g has inverse $R_{g^{-1}}$ where multiplication is smooth by the virtue of G being a Lie group. These maps can be used to get around the whole Lie group namely any $a \in G$ can be moved to $b \in G$ by $L_{a^{-1}b}$ or $R_{ba^{-1}}$.

Moreover such a translation operator in conjunction with the pushforward induce a map $dL_g : T_eG \to T_gG$ (sometimes denoted $(L_g)_*$) where

$$dL_g(\frac{d}{dt}(\gamma(t))\big|_{t=0}) = \frac{d}{dt}(L_g(\gamma(t)))\big|_{t=0}$$

is its explicit form. It is a vector space isomorphism.

Definition 4.4. *A vector field X on a lie group G is a* ***left invariant vector field*** *if*

$$X \circ L_g = (L_g)_* X \text{ for all } g \in G$$

or explicitly

$$X_{gh} = (L_h)|_* X_g \text{ for all } g, h \in G$$

A left invariant vector field on a lie group G has the important property that is is determined by its value at the identity element e of the lie group since

$$X_g = (L_g)_* X_e \text{ for all } a \in G$$

For matrix groups $X_a = L_a \cdot X_e$ the operation can be considered to be matrix multiplication so the notation is justified.

5 One Parameter Subgroups and the Exponential Map

Definition 5.1. *The one parameter subgroup γ of G is defined to be a lie group homomorphism $\gamma : \mathbb{R} \to G$ where $\mathbb{R}$ is the group of additive real numbers.*

We now derive the form of the map of the one parameter subgroup when G is a matrix lie group.

Lemma 5.1. *The if G is a matrix Lie group then our one parameter subgroup $f : \mathbb{R} \to G$ necessarily takes the form of the matrix exponential.*

$$f(t) = e^{tA} \text{ where } A \text{ is the matrix } f'(0).$$

Proof. From the definition of the derivative we have

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{f(t)f(h) - f(t)}{h} \\ &= (\lim_{h \to 0} \frac{f(h) - 1}{h}) f(t) \\ &= f'(0) f(t) \\ &= A f(t) \end{aligned}$$

Then we have the differential equation with initial condition $f(0) = 1$ we have[4]

$$f'(t) = Af(t) \text{ where } f(0) = 1$$

Solving this elementary differential equation we obtain the function

$$f(t) = e^{At+c}$$

then as $f(0) = 1 = \exp(c)$ we must have $c = 0$.

The matrix A is called the **infinitesimal generator** of the subgroup $f(t)$ □

Example 5.1. *If we are given the one parameter subgroup*

$$\gamma(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \text{ of } U(2)$$

then the infinitesimal generator of the subgroup is found by simply taking the derivative.

$$\gamma'(t) = \begin{bmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

So the infinitesimal generator of $\gamma(t)$ is the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which is a rotation by $-\frac{\pi}{2}$
Note that this matrix is not in $U(2)$ but in its tangent space.

Theorem 5.2. *For **any** Lie group G, there is a one to one correspondence between one parameter subgroups of G and the tangent space T_eG.*

Proof. One direction is clear since a homomorphism f gives us a tangent vector $f'(0)$ by differentiation. Now given a vector $v \in T_eG$ We need to find a homomorphism f. We can use the Lie groups translation to generate the left invariant vector field

$$X_g^v = (dL_g)_e(v)$$

Let $f : (-\epsilon, \epsilon) \to G$ be the unique integral curve given of X_g^v such that $f(0) = 1$ and

$$(f_t)_* \frac{d}{dt} = X_{f(t)}^v = (dL_{f(t)})_e(v)$$

This curve is a homomorphism since if we fix a $s \in I$ such that $s + t \in I$ for all $t \in I$ then the curves

$$t \mapsto f(t+s) \text{ and } t \mapsto f(s)f(t)$$

satisfy the previous equation (the second by left invariance) and take the common value $f(s)$ when $t = 0$. Thus by the uniqueness for the solution we obtain

$$f(s+t) = f(s)f(t)$$

We now extend the domain of f to all of $\mathbb{R}$ by noting that the element $f(t/n)^n$ is defined for all large n and is independent of such an n since for an m we have

$$f(\frac{t}{n})^n = f(\frac{t}{nm})^{nm} = f(\frac{t}{m})^m$$

Therefore we can define $f(t) = f(t/n)^n$ for any large n and thus we have obtained the desired homomorphism. □

Remark. The matrix exponential map $\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a local diffeomorphism in a neighbourhood of the zero matrix $0 \in M_n(\mathbb{R})$. Its inverse map is the smooth map given by matrix logarithm $\log : GL_n(\mathbb{R}) \to M_n(\mathbb{R})$ that sends $g \mapsto \log(g)$ when $||g - 1|| < 1$, by

$$\log(1 - A) = -\sum_{k=1}^{\infty} \frac{A^k}{k} \text{ where } A \in GL_n(\mathbb{R})$$

5.1 Examples

Example 5.2. *Recall the Lie Group $O(n)$ is defined as*

$$O(n) = \{O \in GL(n, \mathbb{R}) | O^T O = I\}$$

The tangent space $T_e O(n)$ is equal to the set of $n \times n$-skew symmetric matrices denoted $\mathfrak{o}(n)$.

Proof. Let $A \in T_I O(n)$, then there is a curve $B(t) : J \to O(n) \subset \mathbb{R}^{n^2}$ in $O(n)$ such that $B(0) = I$ and $\dot{B}(0) = A$. Moreover we must have that $B(t)$ satisfies the relation

$$B^T(t)B(t) = I$$

Applying $\frac{d}{dt}\Big|_{t=0}$ we get

$$\dot{B}(0)B(0) + B^T(0)\dot{B}(0) = 0$$

$$A^T \cdot I + I^T \cdot A = 0$$

Then $A = -A^T$ so A is a skew symmetric matrix.
For the converse suppose $A \in \mathfrak{o}(n)$ where $\mathfrak{o}(n)$ is the set of $n \times n$-skew symmetric matrices, then we have

$$B(t) = \exp(tA) = I + tA + \frac{t^2 A^2}{2} + \ldots$$

then

$$B(0) = \exp(0) = I \text{ and } \dot{B}(0) = A$$

with

$$B^T(t)B(t) = \exp\big(tA^T\big) \exp(tA) = \exp\big(t(-A^T)\big) \exp(tA) = \exp(0) = I$$

Consequently $B(t)$ lies in $O(n)$ for all t and hence $T_I O(n)) = \mathfrak{o}(n)$ □

Exercise 5.1. *Prove using a similar approach that $T_I U(n) = \mathfrak{u}(n)$ where*

$$\mathfrak{u}(n) = \{n \times n\text{-skew hermitian matrices }\}$$

Lemma 5.3. *Let $A \in GL_n(\mathbb{C})$ then we have the following identity*

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A))$$

Exercise 5.2. *Prove Lemma 5.3.*
Hint: Any non diagonalizable complex matrix can be expressed as the limit of a sequence of diagonalizable matrices.

Example 5.3. *Recall the lie group $SL_n(\mathbb{R})$ is defined as*

$$SL_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R}) | \det(M) = 1\}$$

The tangent space of $T_e SL_n(\mathbb{R}) = \mathfrak{sl}_n(\mathbb{R})$, where

$$\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_\mathfrak{n}(\mathbb{R}) | \mathrm{Tr}(M) = 0\}$$

Proof. Let $A \in T_e(SL_n(\mathbb{R}))$, then the one parameter subgroup generated by A is $\exp(tA) \in SL_n(\mathbb{R})$ for some t. Therefore

$$\det(\exp(tA)) = 1$$
$$\exp(\mathrm{Tr}(tA)) = 1$$

so $\mathrm{Tr}(A)) = 0$ and thus $A \in \mathfrak{sl}_n(\mathbb{R})$.

For the converse suppose $A \in \mathfrak{sl}_n(\mathbb{R})$ then taking the determinant of the exponential map we obtain

$$\det(\exp(tA)) = \exp(\mathrm{Tr}(A)) = 1$$

therefore $\exp(tA) \in SL_n(\mathbb{R})$ and therefore $A \in T_e SL_n(\mathbb{R})$. □

Example 5.4. *The lie group $SU(n)$ is the set of all unitary matrices with determinant 1. Its tangent space at the identity is the set*

$$\mathfrak{su}(n) = \{n \times n\text{-skew hermitian matrices with trace zero}\}$$

If we consider when $n = 2$ then from the talk given by Eskander Salloum this corresponds to the space of pure quaternions.

$$\bar{\mathbb{H}} = \{ai + bj + ck | a, b, c \in \mathbb{R} \text{ and } i^2 = j^2 = k^2 = -1\}$$

Taking a basis vector u in this space we have that $u^2 = -1$, the exponential map of this element then gives us the one parameter subgroup

$$e^{tu} = \cos(t) + u\sin(t).$$

The subgroup of rotations about u.

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