Recap

A <u>homogeneous space</u> is a space X on which a Lie Group G act transitively.

Let $x_0 \in X$ and H be the isotropy group of x_0.

- orbit $G \cdot x_0 = X$
- prop $G/H \cong X$

eg. $S^{n-1} \cong O_n / O_{n-1}$ $\quad$ $H \cong SL_2\mathbb{R}/SO_2$

$Gr_k(\mathbb{R}^n) \cong O_n / O_k \times O_{n-k}$ $\quad$ $\mathcal{P} \cong GL_n\mathbb{R}/O_n$

Symmetric Spaces

Def A <u>Riemannian manifold</u> is

- a manifold X ← a topological space that locally resembles Euclidean space near each point.

with

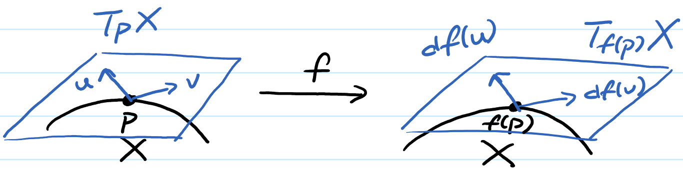
- a Riemannian metric g
  - assigns to each point $p \in X$ a positive-definite inner product $g_p : T_pX \times T_pX \to \mathbb{R}$

$\uparrow$ tangent space of X at p - a vector space

isomorphism of smooth manifolds

Def An <u>isometry</u> of (X, g) is a mapping $f: X \to X$ (diffeomorphism) that preserves the metric.

ie.
$$f : X \longrightarrow X$$
$$p \longmapsto f(p)$$



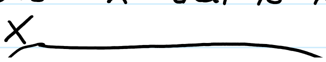
$$df : T_pX \longrightarrow T_{f(p)}X$$
$$u \longmapsto df(u)$$

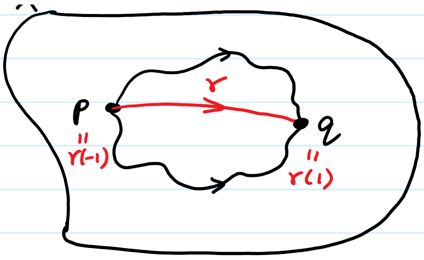
$$g_p(u, v) = g_{f(p)}(df(u), df(v))$$

Def The <u>isometry group</u> of X is $\mathcal{I}(X) = \{ f : X \to X \mid f \text{ is an isometry} \}$.
with function composition

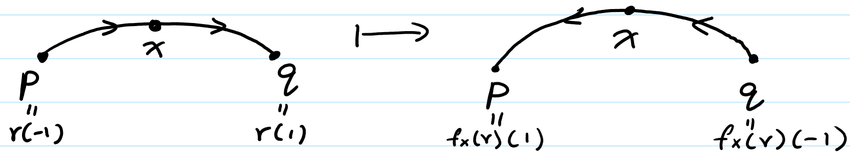
- This is a Lie group.

Def A <u>geodesic</u> on (X, g) is a curve $\gamma : [-1, 1] \to X$ that is length-minimizing.





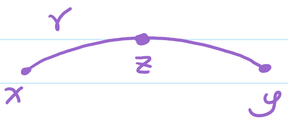
<u>Def</u> (X, g) is a <u>symmetric space</u> if
for each $x \in X$, $\exists$ isometry $f_x : X \to X$ $\in I(X)$ that
reverses geodesics through x.



$$\gamma(t) \longmapsto f_x(\gamma)(-t)$$

<u>Prop</u> A symmetric space (X, g) is homogenous
under the isometry group $I(X)$.

<u>Pf</u> Let $x, y \in X$. (Assume path connectedness)
Find $f \in I(X)$ s.t. $f(x) = y$.



$\exists$ a geodesic γ connecting x and y.
Let z be the mid-point of γ.
Then $f_z(x) = y$.

If we fix a point $x_0 \in X$ and let $H = \{g \in I(X) \mid g x_0 = x_0\}$,
then $X \cong I(X)/H$

<u>EG-0</u> $X = \mathbb{R}^n$ with the euclidean metric
$I(X) = E(n) \leftarrow$ Euclidean group

- $\mathbb{R}^n$ is a symmetric space since :
  At any point $x \in \mathbb{R}^n$, point reflection $f_x(x+v) = x - v \in E(n)$

- Let $x_0 =$ origin.
  Then the isotropy group of x_0 is O_n.

So $\mathbb{R}^n \cong E(n)/O_n$.

Let us still consider our $x_0 \in X$.
Let f_x be the isometry that reverses the geodesics through x_0.
For easier notation, let $G = I(X)$ be the isometry group of X.
Now, let us consider the following automorphism of G :

$$\alpha : G \to G$$
$$g \mapsto f_x \circ g \circ f_x$$

$\leftarrow$ Élie Cartan
complete classification of

$\alpha: G \to G$

$g \mapsto f_x \circ g \circ f_x$

← Élie Cartan
complete classification of order 2 automorphism of the isometry Lie group of Riemannian symmetric spaces.

Note that $\alpha^2(g) = f_x^2 \circ g \circ f_x^2 = g \quad \forall g \in G.$

Let $G^\alpha = \{g \in G \mid \alpha(g) = g\}$ be the subgroup of G left fix by α.
G^α may contain multiple connected components. Let us denote the connected component containing the identity (ie <u>identity component</u>) as G^α_0.

<u>Proposition</u> $\quad G^\alpha_0 \subseteq H \subseteq G^\alpha$

<u>EG-0</u> Riemannian space $\mathbb{R}^n$ with isometry group $G = E(n)$

Let x_0 still be the origin.
So $f_x(v) = -v$.

$$\Rightarrow \alpha(g)(v) = f_x \circ g \circ f_x(v) = f_x \circ g(-v) = -g(-v)$$

Here, $G^\alpha = O_n$ since:

($\Rightarrow$)
$-g(-v) = g(v) \quad \forall v \in \mathbb{R}^n$
$\Rightarrow -g(0) = g(0)$
$g(0) = 0$
g is an isometry that fixes the origin
$\Rightarrow g \in O_n$

($\Leftarrow$)
$g \in O_n$ is linear
$\Rightarrow -g(-v) = g(-(-v)) = g(v)$

So $O_n = \{g \in E(n) \mid \alpha(g) = g\} = G^\alpha$.

Recall that the isotropy group of x_0 is O_n, and the identity component of $G^\alpha = O_n$ is SO_n. So here we indeed have the isotropy group in between $G^\alpha_0 = SO_n$ and $G^\alpha = O_n$.

In fact here, G^α happens to equal to the isotropy group. But this doesn't always happen.

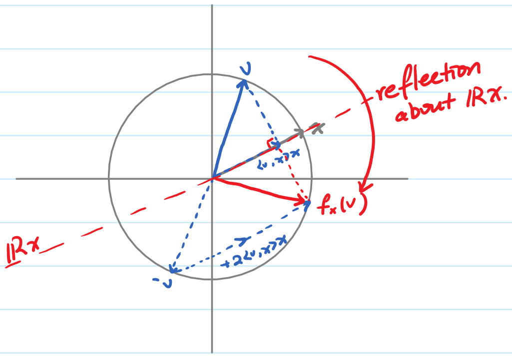
Recall our earlier examples of homogeneous spaces:
- the sphere S^{n-1} homogeneous under O_n
- the upper half-plane $\mathcal{H}$ under $SL_2\mathbb{R}$
- the set of symmetric, positive definite, real matrices $\mathcal{P}$ under $GL_n\mathbb{R}$
- the Grassmannian $Gr_k(\mathbb{R}^n)$ under O_n

These 4 spaces are all symmetric spaces as well.

EG-1 S^{n-1} under Lie group $G = O_n$.

S^{n-1} is a symmetric space since for any $x \in S^{n-1}$, there exists an isometry f_x being the reflection in the line $\mathbb{R}x$.
ie. $f_x(v) = -v + 2\langle v, x\rangle x$

For instance in S^1:



Fix $x_0 = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$.

Then $f_x\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix} = \begin{pmatrix}-v_1\\-v_2\\\vdots\\-v_n\end{pmatrix} + \begin{pmatrix}2v_1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}v_1\\-v_2\\\vdots\\-v_n\end{pmatrix}$

So f_x with respect to the standard basis is $\begin{bmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots & \\ & & & & -1\end{bmatrix}$.

Consider $\alpha : O_n \to O_n$
$g \mapsto f_x g f_x$.

$G^\alpha = O_{n-1} \times O_1$ since:

$$\underbrace{\left[\begin{array}{c|c}1 & 0\\ \hline 0 & -I_{n-1}\end{array}\right]}_{f_x}\underbrace{\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]}_{g}\underbrace{\left[\begin{array}{c|c}1 & 0\\ \hline 0 & -I_{n-1}\end{array}\right]}_{f_x} = \underbrace{\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]}_{g}$$

$$\iff \left[\begin{array}{c|c}A & -B\\ \hline -C & D\end{array}\right] = \left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]$$

$\iff$ $B = C = 0$ and $\underbrace{A = \pm 1,\ D \in O_{n-1}}_{\text{to ensure } g \in O_n}$

Note that G^α contains 4 connected components:

$O_{n-1} \times O_1 = SO_{n-1} \times \{+1\} \cup SO_{n-1} \times \{-1\}$
$\cup\ (O_{n-1} \setminus SO_{n-1}) \times \{+1\} \cup (O_{n-1} \setminus SO_{n-1}) \times \{-1\}$.

The identity component G_0^α is $\{SO_{n-1}\} \times \{+1\} \cong SO_{n-1}$.

Recall that the isotropy group of $x_0 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ is $H = O_{n-1}$.

Indeed we have $SO_2 \leq O_{n-1} \leq O_{n-1} \times O_1$
ie. $G_0^\alpha \leq H \leq G^\alpha$

Complex structures on $\mathbb{R}^{2n}$

Def A complex structure on $\mathbb{R}^{2n}$ is a linear transformation $J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -1$.

Let $\mathcal{S}_n$ be the space of complex structures on $\mathbb{R}^{2n}$ that preserves the inner product.

ie. if $J \in \mathcal{S}_n$, then $(J\vec{u}, J\vec{v}) = (\vec{u}, \vec{v})$ $\forall u, v \in \mathbb{R}^{2n}$

ie $\mathcal{S}_n = \{ J \in O_{2n} \mid J^2 = -1 \}$

Lemma Let $\vec{u} \in \mathbb{R}^{2n}$, $J \in \mathcal{S}_n$. Then $(\vec{u}, J\vec{u}) = 0$

Proof
$$\begin{aligned}(\vec{u}, J\vec{u}) &= (J\vec{u}, J^2\vec{u}) \\ &= (J\vec{u}, -\vec{u}) \\ &= -(J\vec{u}, \vec{u}) \\ &= -(\vec{u}, J\vec{u})\end{aligned}$$
$\therefore\ 2(\vec{u}, J\vec{u}) = 0$

$\Rightarrow\ (\vec{u}, J\vec{u}) = 0$

Prop Let $J \in \mathcal{S}_n$. Then $\exists$ orthonormal basis $\{v_i\}$ of $\mathbb{R}^{2n}$ such that
$$Jv_{2k-1} = v_{2k}, \quad Jv_{2k} = -v_{2k-1}$$

ie. $\exists$ orthonormal $\{v_i\}$ s.t.
$$J = P \begin{bmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{bmatrix} P^{-1} \quad \text{where } P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_{2n} \\ | & | & & | \end{bmatrix}$$

Proof Pick $\vec{v_1} \in \mathbb{R}^{2n}$ s.t. $\vec{v_1} \neq 0$ and $\|\vec{v_1}\| = 1$.
Let $\vec{v_2} = J\vec{v_1}$

$V = \text{span}(\vec{v_1}, \vec{v_2})$ is a subspace of $\mathbb{R}^{2n}$.
We can split $\mathbb{R}^{2n}$ as
$$\mathbb{R}^{2n} = V \oplus V^{\perp}$$

Then pick $\vec{v_3}$ and $\vec{v_4} = J\vec{v_3}$ from $V^{\perp}$.
$\vdots$

$\{ \vec{v_1}, \underset{J\vec{v_1}}{\underset{\parallel}{\vec{v_2}}}, \dots, \vec{v_{2n-1}}, \underset{J\vec{v_{2n-1}}}{\underset{\parallel}{\vec{v_n}}} \}$ is an orthonormal basis

since $(\vec{v_{2k-1}}, \vec{v_{2k}}) = (\vec{v_{2k-1}}, J\vec{v_{2k-1}}) = 0$ (from previous lemma)

With respect to this basis, we obtain J as
$$\begin{bmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & -1 \end{matrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{smallmatrix}1&0\end{smallmatrix} & & & \\ & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\end{bmatrix}$$

As a consequence, any two J's are conjugate in O_{2n}, since:

Let $J_1, J_2 \in \mathcal{J}$.

Then $J_1 = PAP^{-1}$ where $A = \begin{bmatrix} \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\end{bmatrix}$, $P, Q \in O_{2n}$.

$J_2 = QAQ^{-1}$

$$\Rightarrow J_1 = PQ^{-1}J_2QP^{-1}$$
$$= (PQ^{-1})J_2(PQ^{-1})^{-1}$$

So $\mathcal{J}_n$ is homogenous under conjugation action by O_{2n}.

Prop $\mathcal{J}_n \cong O_{2n}/U_n$

Proof:

Let $J_0 = \begin{bmatrix} \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & \\ & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\end{bmatrix}$

It is sufficient to show $U_n = \{g \in O_{2n} \mid g \cdot J_0 = gJ_0g^{-1} = J_0\}$

- Consider the embedding

$$\phi : GL_n(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{R})$$

mapping each complex entry $a+ib$ to $\begin{matrix} a & -b \\ b & a \end{matrix}$ 2x2 block

- ϕ is a homomorphism since

$$(a+ib)(c+id) = (ac-bd) + (ad+bc)i$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix}$$

$$\Rightarrow \phi(AB) = \phi(A)\phi(B)$$

- ϕ is injective since $\ker \phi = I$.

- $GL_n(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{R}) \mid AJ_0A^{-1} = J_0\}$

since: $AJ_0 = J_0A$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c & -a \\ d & -b \end{bmatrix} = \begin{bmatrix} -b & -d \\ a & c \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} c & -a \\ d & -b \end{bmatrix} = \begin{bmatrix} -b & -d \\ a & c \end{bmatrix}$$

$$\Leftrightarrow \quad c = -b, \; d = a$$

$\square$ $U_n = \{ g \in O_{2n} \mid g \cdot J_0 = gJg^{-1} = J_0 \}$

$= O_{2n} \cap \{ g \in GL_{2n}(\mathbb{R}) \mid AJ_0A^{-1} = J_0 \}$
$= O_{2n} \cap GL_n(\mathbb{C})$
$= \{ A \in GL_n(\mathbb{C}) \mid \phi(A) \in O_{2n} \}$

$u^{-1} = u^*$
$\Leftrightarrow \phi(u^{-1}) = \phi(u^*)$ (ϕ injective)
$\Leftrightarrow \phi(u)^{-1} = \phi(u)^T$ (ϕ homomorphism and $\phi(a - ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$)
$\Leftrightarrow \phi(u) \in O_{2n}$

Therefore, $U_n = O_{2n} \cap GL_n(\mathbb{C})$
is the isotropy group of J_0.

$\therefore \; \mathcal{J}_n \cong O_{2n}/U_n$

<u>Def</u> the isotropic Grassmannian $Gr_n^{iso}(\mathbb{C}^{2n})$
$= \{$ n-dimensional complex subspace $W \leq \mathbb{C}^{2n}$
$\mid W = W^\perp \}$
$(w, w') = 0 \; \forall w \in W$ where $(\,,\,)$
is the $\mathbb{C}$-bilinear extension of the
inner product of $\mathbb{R}^{2n}$. $(\vec{u}, c\vec{v}) = c(\vec{u}, \vec{v}) \; \forall c \in \mathbb{C}$.

<u>Prop</u> $\mathcal{J}_n \cong Gr_n^{iso}(\mathbb{C}^{2n})$
$J \longmapsto W = i$-eigenspace of J