$SU_2 \to SO_3$

1) definition of SU_2

$$SU_2 = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \ a, b \in \mathbb{C}, \ a\bar{a} + b\bar{b} = 1 \right\}$$

excersize derive this

letting $a = x_0 + x_1 i$, $b = x_2 + x_3 i$

$$a\bar{a} + b\bar{b} = x_0^2 + x_1^2 + x_2^2 + x_3^2$$
$$= 1$$

So SU_2 topologically is $S^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$

2) definition of quaterions and unit quaterions.

$$\mathbb{H} = \{ x + y\hat{i} + z\hat{j} + w\hat{k} \mid x, y, z, w \in \mathbb{R} \text{ and } i^2 = j^2 = k^2 = -1 \text{ and } ijk = -1, \text{ so } ij = k, \ jk = i, \ ki = j \}$$

$$S\mathbb{H} = \{ x + y\hat{i} + z\hat{j} + w\hat{k} \mid x^2 + y^2 + z^2 + w^2 = 1 \text{ and all conditions from } \mathbb{H} \}$$

3) isomorphism between $S\mathbb{H}$ and SU_2

$$U_1(\mathbb{H}): \quad S\mathbb{H} \longrightarrow SU_2$$
$$(x, y, z, w) \longmapsto \begin{pmatrix} x + yi & z + wi \\ -z + wi & x - yi \end{pmatrix}$$

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\hat{i} \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\hat{j} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\hat{k} \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is a group homomorphism excersize.

4) isomorphism between $\mathbb{R}^3$ and $su(2)$

where $su(2) = \{ A \in M_{2\times 2}(\mathbb{C}) \mid tr(A) = 0, \ A = -\bar{A}^T \}$

so if ~~[illegible]~~ $A \in su(2)$ $\quad A = \begin{pmatrix} ai & b + ci \\ -b + ci & -ai \end{pmatrix}$, $a, b, c \in \mathbb{R}$

$\Phi : \mathbb{R}^3 \longrightarrow su(2)$

$(a, b, c) \longmapsto \begin{bmatrix} ai & b+ci \\ -b+ci & -ai \end{bmatrix}$

group homomorphism under addition, excersize check this.

$\hat{i} = (1,0,0) \longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\hat{j} = (0,1,0) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\hat{k} = (0,0,1) \longmapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

5) define the rotation map

let $T_g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\vec{v} \longmapsto \Phi^{-1}(g \cdot \Phi(\vec{v}) \cdot g^{-1})$

with $g \in SU_2$ a 2×2 matrix

to check this works we need to check

A) ~~g^{-1}~~ $g \cdot (\Phi(\vec{v})) \cdot g^{-1} \in su(2)$

so $tr(g \cdot \Phi(\vec{v}) \cdot g^{-1}) = 0$

and $g \cdot \Phi(\vec{v}) \cdot g^{-1} = -\overline{(g \cdot \Phi(\vec{v}) \cdot g^{-1})}^T$

these are excersizes

B) we also need to check this is a rotation to do this we use Euler's ~~theorem~~ theorem on Page 133 artin which say if T_g is a linear transformation which is orthogonal and determinant 1 then T_g is in SO_3.

- Showing T_g is a linear transformation is an excersize
- to show orthogonality we use the identity

$$\langle \vec{u}, \vec{v} \rangle = -\frac{1}{2} tr(\Phi(\vec{u}), \Phi(\vec{v})), \text{ for } \vec{u}, \vec{v} \in \mathbb{R}^3$$

Showing this identity is an excersize

to show orthogonality is to show $\langle T_g \vec{u}, T_g \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle$

$$\langle T_g \vec{u}, T_g \vec{v} \rangle = -\frac{1}{2} tr(\Phi(T_g \vec{u}) \cdot \Phi(T_g \vec{v}))$$

$$= -\frac{1}{2} tr(\Phi(g \cdot \Phi(\vec{u}) \cdot g^{-1} \cdot g \cdot \Phi(\vec{v}) \cdot g^{-1}))$$

$$= -\frac{1}{2} tr(\Phi(g \cdot \Phi(\vec{u}) \cdot \Phi(\vec{v}) \cdot g^{-1}))$$

Because conjugation preserves trace, excersize

$$= -\frac{1}{2} tr(\Phi(\vec{u}) \cdot \Phi(\vec{v}))$$

$$= \langle \vec{u}, \vec{v} \rangle$$

So T_g is orthogonal.

- to show determinant is 1, first note that the determinant of orthogonal matrices is ± 1 then since the sphere is path connected and the determinant is continuous then $\det(T_g) = \pm 1$ but not both. Using the identity element the determinant is 1.

So by Euler's theorem all T_g are rotations SO_3

6) Now defining a map to SO_3

$$T: SU_2 \longrightarrow SO_3$$
$$g \longmapsto T_g$$

we know $T_g \in SO_3$ ~~[illegible]~~

we still need to show surjectivity. To do this we need to define the angle of rotation for a given $g \in SU_2$.

First we write $g \in SU_2$ as $g = \cos(\theta) I + \sin(\theta) A$

with $A \in su(2)$ so that $\det(A) = 1$ checking this expression g is in SU_2 is an exercise.

Now we want to show for any $g \in SU_2$ that the rotation which $g = \cos(\theta) \underline{I} + \sin(\theta) A$ effects is exactly 2θ.

To do this note that a rotation in any direction will be symmetric and we can use any g to show this because it is a rotation. [exercise Artin page 272]

So chose $g = \cos(\theta)\, \underline{I} + \sin(\theta)\, \Phi(\hat{i}) = \begin{bmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{bmatrix} + \begin{bmatrix} \sin(\theta) i & 0 \\ 0 & -\sin(\theta) i \end{bmatrix}$

$$= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

and choose $\vec{v} = \hat{j} = (0,1,0)$

so $g \cdot \Phi(\vec{v}) \cdot g^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 0 & e^{2i\theta} \\ -e^{-2i\theta} & 0 \end{bmatrix}$

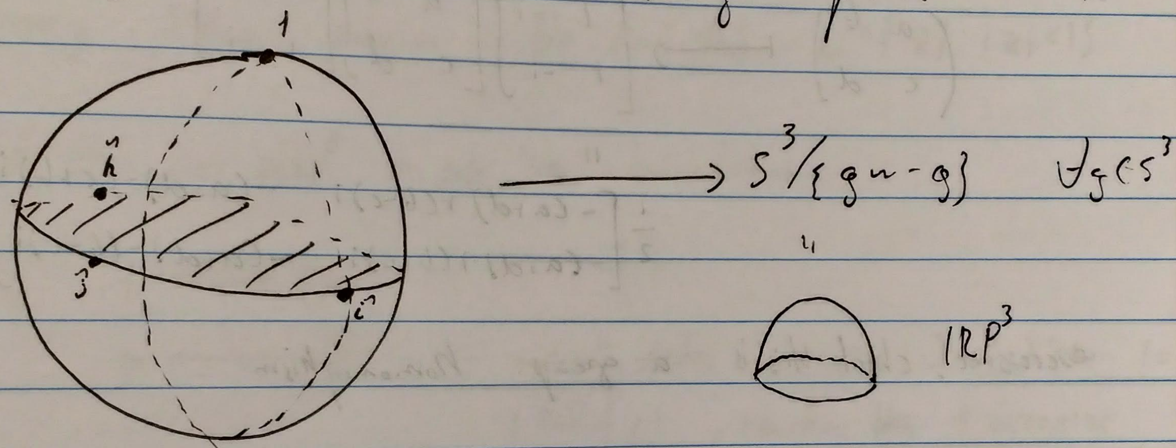
$$= \cos(2\theta)\, \hat{j} + \sin(2\theta)\, \hat{k}$$

which shows the angle of rotation is 2θ. Also that the resultant vector was in the $\hat{j}, \hat{k}$ direction which is orthogonal to $\hat{i}$ as expected.

So we can say T is surjective.

As $T_g = T_{-g}$ the T is also a double cover

7) visualizing this a little we can try to picture $S^3 \subseteq \mathbb{R}^4$



- i, j, k are on the equator as their trace is 0.
- the equator is the set of points with trace 0, skew-hermitian which is all points from $\mathbb{R}^3$
- note all points in diagram have $\det(A)=1$ as in SU_2
- the longitudes of $\hat{i}, \hat{j}, \hat{k}$ are rotations in the $\hat{i}, \hat{j}, \hat{k}$ directions respectively.

Picturing $SL_2(\mathbb{R})$

1) definition $SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \det(A)=1 \ , \ x,y,z,w \in \mathbb{R} \right\}$

2) definition $SU_{1,1} = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \middle| |a|^2 - |b|^2 = 1 \ , \ a,b \in \mathbb{C} \right\}$

3) making an isomorphism between $SL_2(\mathbb{R})$ and $SU_{1,1}$

$$L : SL_2(\mathbb{R}) \longrightarrow SU_{1,1} \quad , \quad \xi = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} -(a+d)+(b-c)i & -(a-d)-(c+b)i \\ -(a+d)+(b+c)i & -(a+d)-(b-c)i \end{bmatrix}$$

excersize, check this is a group homomorphism

4) showing $SL_2(\mathbb{R})$ preserves the upper half plane

letting the upper half plane be $H = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

to show: if $z \in H$ then $\frac{az+b}{cz+d} \in H$ where $\frac{az+b}{cz+d}$ is the mobius transform, showing mobius transform is well defined is an excersize.

let $z = x + yi$, $y > 0$

$$z \longmapsto \frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}} = \frac{(ax+ayi+b)\overline{(cx+cyi+d)}}{(cx+cyi+d)\overline{(cx+cyi+d)}}$$

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{-axcy-bcy+acyx+ady}{[(cx+d+cyi)(cx+d-cyi)]} = \gamma > 0$$

$$= \frac{y(ad-bc)}{\gamma}$$

$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{y}{\gamma}$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $y > 0$ as $Im(z) > 0$

$= y/\gamma > 0$

so $SL_2(\mathbb{R})$ preserves the upperhalf plane.

5) showing $SU_{1,1}$ preserves the unit disk $D = \{z \mid |z| < 1\}$

to do this we need to use the transformation

$$Q : H \longrightarrow D$$
$$z \longmapsto \frac{z-i}{z+i}$$

which takes z with $Im(z) > 0$ to $\left|\frac{z-i}{z+i}\right| < 1$, checking this is excersize

using the above map we can see that

$$P : D \longrightarrow D$$
$$z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

with $|z| < 1$ where

$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = g \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot g^{-1}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$

as $SL_2(\mathbb{R})$ presesves H and g maps H to D. so g^{-1} maps D to H.

6) showing $SU_{1,1}$ is homeomorphic to an open solid torus $S^1 \times D$

using the mapping $R: \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \longmapsto \left(\frac{a}{|a|}, b/a \right)$

we see $\frac{a}{|a|} \in S^1$ as $\left| \frac{a}{|a|} \right| = 1 \quad \forall a$

and $\frac{b}{a} \in D$ as $\left| \frac{b}{a} \right|^2 = \frac{|b|^2}{|a|^2} = \frac{|b|^2}{1+|b|^2} < 1 \quad \forall b$

so $SU_{1,1}$ is homeomorphic to the open solid torus

7) different sections of the torus,

- the centre ± 1
- hyperbolic elements, $|tr| > 2$
- elliptic elements, $|tr| < 2$
- parabolic elements, $|tr| = 2$

identifying these sections, let $A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in SU_{1,1}$

then $\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -\bar{b} & \lambda - \bar{a} \end{vmatrix}$

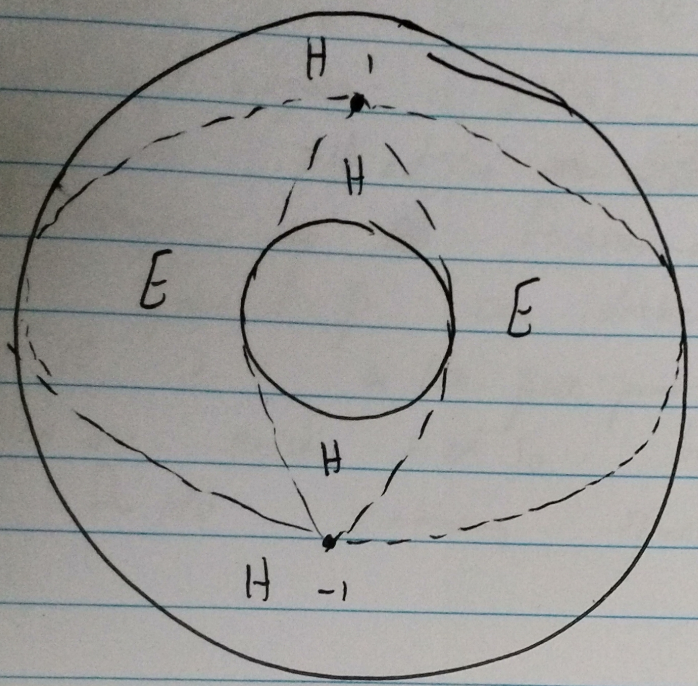
$$= \lambda^2 + a^2 - \bar{a}\lambda - a\lambda - b^2$$

$$= \lambda^2 - tr(A)\lambda + 1$$

$$\Rightarrow \lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4}}{2}$$

looking at the discriminant we see hyperbolic elements have complex eigenvalues, elliptic with real eigenvalues and Parrabolic with eigenvalues ± 1

we see that there are each as below



and the dotted boundary is the parrabolic area.