Homogeneous spaces

Wednesday, 25 August 2021

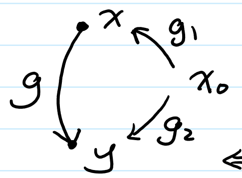
Def A <u>homogeneous space</u> is a space X on which a Lie group G acts transitively.
ie. if $x, y \in X$, then $\exists g \in G$ s.t. $gx = y$.

← Lie groups arise as transformation groups.

<u>EG-1</u> $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is homogeneous under Lie group $O_n = \{g \in GL_n(\mathbb{R}) \mid \langle gx, gy \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{R}^n\}$
$= \{A \in GL_n(\mathbb{R}) \mid AA^T = I\}$

- group action $O_n \times S^{n-1} \to S^{n-1}$ is defined:
  if $g \in O_n$, $x \in S^{n-1}$, then $\|gx\| = \sqrt{\langle gx, gx \rangle} = \sqrt{\langle x, x \rangle} = \|x\| = 1$.
  $\Rightarrow gx \in S^{n-1}$

- O_n acts transitively on S^{n-1}



$g = g_2 g_1^{-1}$

ie G acts transitively on X
$\Leftrightarrow$ Given $x_0 \in X$, if $y \in X$, then $\exists g \in G$ s.t. $gx_0 = y$.

Let $x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ } n rows $\in S^{n-1} \subseteq \mathbb{R}^n$.

Let $y \in S^{n-1}$.

Find $g \in O_n$ s.t. $gx_0 = y$

$$\begin{bmatrix} | & | & & | \\ y & b_1 & \cdots & b_{n-1} \\ | & | & & | \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} | \\ y \\ | \end{pmatrix}$$

y needs to be in the first column

Let $W^\perp \leq \mathbb{R}^n$ be orthogonal to $W = \langle y \rangle$.
Find an orthonormal basis of $W^\perp$ via Gram-Schmidt: $(b_1, b_2, \ldots, b_{n-1})$.

← $\langle\ ,\ \rangle$ in $\mathbb{R}^n$ is symmetric and non-isotropic $\langle v, v \rangle = 0 \Rightarrow v = 0$

O_n: $AA^T = I \Leftrightarrow$ columns of A are orthonormal

$y, b_1, \ldots, b_{n-1}$ are orthonormal $\Rightarrow \in O_n$.

<u>EG-2</u> $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is homogenous under $SL_2\mathbb{R}$ via action

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z \right) \longmapsto \frac{az + b}{cz + d}$$

[We have checked this action is defined last week with Esky.]

- Let $x_0 = i$.
  Let $z = a+bi \in H$, $b>0$.

  Find $g \in SL_2\mathbb{R}$ s.t. $gi = a+bi$

  Try $g = \begin{bmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{bmatrix}$ $(b>0)$ ← motivation: $\begin{cases} a+bi = \dfrac{mi+n}{li+k} \\ mk - ln = 1 \end{cases}$ and set $l=0$.

  - $\dfrac{\sqrt{b}\, i + a/\sqrt{b}}{1/\sqrt{b}} = bi + a$

  - $\det(g) = 1 \Rightarrow g \in SL_2\mathbb{R}$.

  So $SL_2\mathbb{R}$ acts transitively on H.

EG-3 $\mathcal{P} = \{P \in M_n(\mathbb{R}) \mid P$ is symmetric and positive definite$\}$ is homogenous under $GL_n\mathbb{R}$ via action
$(A, P) \longmapsto APA^t$

- well-defined
  - symmetric $(APA^t)^t = AP^tA^t = APA^t$
  - positive definite $x(APA^t)x^t = (xA)P(xA)^t > 0$ $\forall x \in \mathbb{R}^n$ row vectors.

- Transitive
  Let $P_0 = I$ and $P \in \mathcal{P}$.
  P is symmetric $\Rightarrow P = UDU^{-1}$ for some orthogonal U and diagonal $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$

  P is also positive-definite $\Rightarrow d_i \geq 0$

  Let $D' = \begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix}$.

  $$\begin{aligned} P &= UDU^T \\ &= UD'D'U^T \quad (U^T = U^{-1}) \\ &= (UD')(UD')^T \\ &= \underset{A}{(UD')}\ \underset{P_0}{I}\ \underset{A^T}{(UD')^T} \end{aligned}$$

EG-4 Grassmannian $Gr_k(\mathbb{R}^n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

$Gr_k(\mathbb{R}^n)$ is homogenous under $GL_n\mathbb{R}$.

- closed under group action.
  W: k-dimensional subspace with basis $(b_1, b_2, \ldots, b_k)$.
  Then
  $(g, \; w = a_1b_1 + \cdots + a_kb_k) \mapsto$
  $gw = a_1gb_1 + \cdots + a_kgb_k$

gW is still a k-dimentional subspace with basis $(gb_1, \dots, gb_k)$.

- Let W_0 be generated by $(e_1, \dots, e_k)$. Let $W \in Gr_k(\mathbb{R}^n)$ be generated by $(y_1, \dots, y_k)$.

  Find $g \in GL_n\mathbb{R}$ s.t. $gW_0 = W$.

$$\underbrace{\begin{bmatrix} | & & | & | & & | \\ y_1 & \cdots & y_k & y_{k+1} & \cdots & y_n \\ | & & | & | & & | \end{bmatrix}}_{GL_n\mathbb{R}} \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & 1 \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 0 \end{bmatrix} = \begin{bmatrix} | & & | \\ y_1 & \cdots & y_k \\ | & & | \end{bmatrix}$$

($y_{k+1}, \dots, y_n$: a basis of $W^\perp$ ← $(\cdot,\cdot)$ is symmetric and non-isotropic)

[$Gr_k(\mathbb{R}^n)$ is also homogenous under O_n – we can always find orthonormal y_i's] → identify X with G/H

Let G be a group that act transitively on a set X, $x_0 \in X$, and H be the stabiliser/isotropy group of x_0 $\{g \in G \mid gx_0 = x_0\}$.

Prop $G/H \xrightarrow{\cong} X$ (← set of left cosets)
$gH \mapsto gx_0$

Proof Let $\varphi: G/H \to X$
$gH \mapsto gx_0$

- φ is well defined since
  $g_1H = g_2H \Rightarrow g_1 = g_2h$ for some $h \in H$
  $\Rightarrow \varphi(g_1H) = g_1x_0 = (g_2h)x_0$
  $= g_2(hx_0)$
  $= g_2x_0$
  $= \varphi(g_2H)$

- φ is injective since
  $\varphi(g_1H) = \varphi(g_2H)$
  $\Rightarrow g_1x_0 = g_2x_0$
  $\Rightarrow g_2^{-1}g_1x_0 = x_0$
  $\Rightarrow g_2^{-1}g_1 \in H$
  $\Rightarrow g_1H = g_2H$

- φ is surjective since
  If $x \in X$, then $\exists g \in G$ s.t. $gx_0 = x$ (G acts transitively)
  $\therefore \varphi(gH) = gx_0 = x$.

EG-1 S^{n-1} homogenous under O_n

$\therefore \ \ \tau(gH) = gx_0 = x.$

EG-1 S^{n-1} homogenous under O_n

$\longrightarrow \quad S^{n-1} \cong O_n / O_{n-1}$ $\quad \leftarrow \dim O_n = \dim O_{n-1} + \underbrace{\dim S^{n-1}}_{n-1}$

Let $x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in S^{n-1}$

To show: $O_{n-1} \cong \{ g \in O_n \mid gx_0 = x_0 \}$

$\dim O_1 = 0$

$\dim O_2 = 0 + 1$

$\vdots$

$\dim O_n = (n-1) + \dim O_{n-1}$

$= (n-1) + (n-2) + \cdots + 1$

$= \frac{n(n-1)}{2}$

$gx_0 = x_0$ $\quad$ ($g \in O_n$ so columns are orthonormal)

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & | & & & | \\ \vdots & b_1 & \cdots & \cdots & b_{n-1} \\ 0 & | & & & | \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

first column is x_0 $\qquad$ b_i's are orthonormal

$$\therefore \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & | & & | \\ \vdots & b_1 & \cdots & b_{n-1} \\ 0 & | & & | \end{bmatrix} \;\middle|\; \langle b_i, b_j \rangle = \delta_{ij} \right\} \cong O_{n-1}$$

EG-2 $H \cong SL_2\mathbb{R} / SO_2$

Let $x_0 = i$.

To show: $\{ g \in SL_2\mathbb{R} \mid gi = i \} = SO_2$

$$\frac{ai+b}{ci+d} = i$$

$\Leftrightarrow ai + b = -c + di$

$$\Leftrightarrow \begin{cases} b = -c \\ a = d \\ ad - bc = 1 \end{cases}$$

$a^2 + b^2 = 1$

Let $a = \cos\theta$, $b = -\sin\theta$ for some $\theta \in [0, 2\pi)$

$\therefore \ c = \sin\theta, \ d = \cos\theta$.

$$\therefore \{ g \in SL_2\mathbb{R} \mid gi = i \} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\} = SO_2$$

EG-3 $\mathcal{P} \cong GL_n\mathbb{R} / O_n$

Let $x_0 = I$.

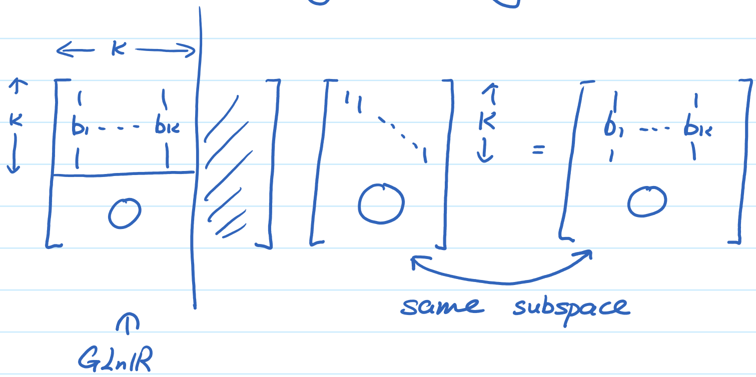
$\{ g \in GL_n\mathbb{R} \mid gIg^t = I \} = O_n$

EG-4 $Gr_k(\mathbb{R}^n) \cong GL_n\mathbb{R} / GL_{k,n-k}$

Let W_0 be generated by $e_1, \ldots, e_k$.

$\leftarrow k \rightarrow |$

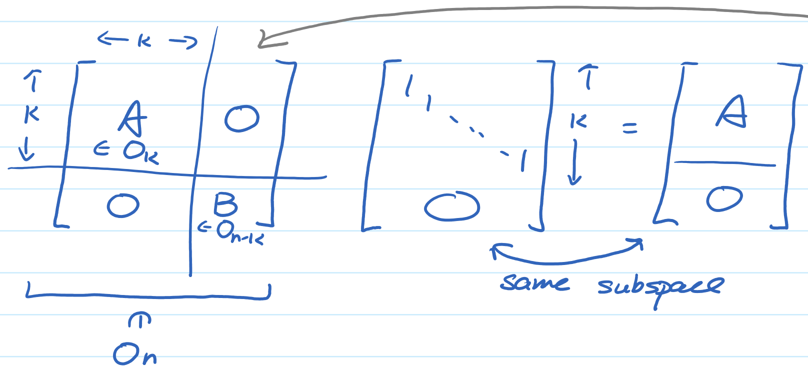
Let W_0 be generated by $e_1, \dots, e_k$.



so columns are linearly independent
so $b_1, \dots, b_k$ are linearly independent.

So $\{g \in GL_n\mathbb{R} \mid gW_0 = W_0\} = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in GL_n\mathbb{R} \right\} = GL_{k,n-k}$

<u>EG-4.2</u> $\quad Gr_k(\mathbb{R}^n) \cong O_n / O_k \times O_{n-k}$



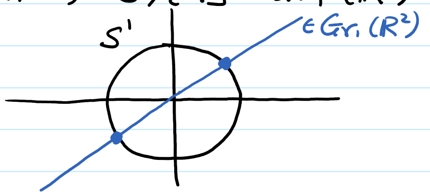
$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} A^t & 0 \\ C^t & B^t \end{bmatrix} = \begin{bmatrix} AA^t + CC^t & CB^t \\ BC^t & BB^t \end{bmatrix} = I$$

$\Rightarrow B = 0$ or $C = 0$

columns are orthonormal $\Rightarrow A \in O_k$ and $B \in O_{n-k}$

<u>RmK</u> Recall $S^{n-1} \cong O_n / O_{n-1}$.
Also, $Gr_{n-1}(\mathbb{R}^n) = O_n / O_{n-1} \times O_1$.
$\Rightarrow \quad S^{n-1}/O_1 = Gr_{n-1}(\mathbb{R}^n)$
$S^{n-1}/\{\pm 1\} = Gr_{n-1}(\mathbb{R}^n)$

When $n=2$, $S^1/\{\pm 1\} = Gr_1(\mathbb{R}^2)$



These 4 isomorphisms are also homeomorphisms between the natural topology of X and G/H as a quotient space of G.