Representation Theory - Student Seminar
Degenerate Affine Hecke Algebras and their Representations

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1 Degenerate Affine Hecke Algebra and Representations

Preamble Hello everyone, today my talk titled 'Degenerate Affine Hecke Algebras and their Representations' will be a continuation from where Weiying Guo's talk left off at the end of chapter 2. That is Weiying covered chapters 1 and 2 to Okounkov and Vershik's paper (**A NEW APPROACH TO THE REPRESENTATION THEORY OF THE SYMMETRIC GROUPS. II**) and I hope to cover chapters 3 & 4. The style of this talk will be to try follow the approach in the paper closely stating the results (mostly) in order of OV and only assuming results from the prior chapters. Let us begin.

We left off in chapter 2 by introducing the set $\mathrm{Spec}(n)$, of which we wish to describe, we now take a slight detour in an effort to help derive an important result (4.1) regarding the vectors $\alpha(v)$ of $\mathrm{Spec}(n)$.

Definition 1.1. In the symmetric group S_n, the adjacent transpositions/2-cycles

$$s_i = (i \ \ i+1), \text{ where } i \in \{1, .., n-1\}$$

are the **Coxeter generators** and generate S_n.

The Coxeter generators satisfy the following relations:

- $s_i^2 = 1$ (Involutive)
- $s_i s_j = s_j s_i$ where $|i-j| \neq 1$ (Commutative)
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (Braid Relation)

Showing the Coxeter Generators Generate S_2 and S_3

$$S_2 = \{(12), (12)(12) = e\}$$
$$S_3 = \{e, (12), (23), (12)(23) = (123), (23)(12) = (132), (12)(23)(12) = (13)\}$$

and so on....

From the relation (ii) in the above we see that Coxeter generators commute except for neighbours. Okounkov refers to such generators as *local* . He then goes on to say 'Locality is meant here as it is understood in physics; it means that the remote generators commute and hence do not effect each other.' The Hamiltonian with nearest neighbour interaction in 1D-Ising model might be an example of what Okounkov means here but going into more detail is unnecessary for the talk at hand.

The sense of locality manifests itself in the following property of the young basis and is captured by the following proposition:

Proposition 1.1. *For any vector in our GZ-basis.*

$$v_T \text{ where } T = \lambda_0 \nearrow \ldots \nearrow \lambda_n, \quad \lambda_i \in S_i^{\wedge}$$

and any $k = 1, \ldots, n-1$*, the vector*

$$s_k \cdot v_T$$

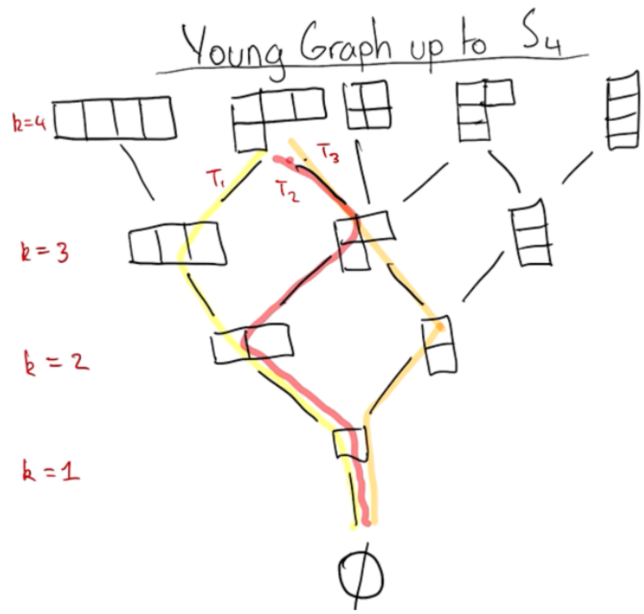
is a linear combination of the vectors

$$v_{T'} \text{ where } T' = \lambda'_0 \nearrow \ldots \nearrow \lambda'_n, \quad \lambda'_i \in S_i^{\wedge}$$

such that

$$\lambda'_i = \lambda_i, \quad i \neq k$$

We now give a concrete example of what this proposition claims before moving to the proof. Consider the following Young Graph: We have



Now the proposition tells us the following.

Fix an equivalence class of irreps of S_4 (second node from the left in this case). Then let $k = 2$ and $T = T_2$

$$s_2 \cdot v_{T_2} = z_1 v_{T_2} + z_2 v_{T_3} \text{ where } z_1, z_2 \in \mathbb{C}$$

as $\lambda_i^{T_2} = \lambda_i^{T_1}$ for $i \neq 2$

Similarly

$$s_2 \cdot v_{T_3} = z_3 v_{T_2} + z_2 v_{T_4} \text{ where } z_3, z_4 \in \mathbb{C}$$

and

$$s_2 \cdot v_{T_1} = z_5 v_{T_1} \text{ where } z_5 \in \mathbb{C}$$

The proof of the proposition is given by the following

Proof. Fix an (equivalence class) irreducible representation V^{λ_n}
Case I) Let $i > k$.
Since $s_k \in S_i$ and the module

$$\mathbb{C}[S_i] \cdot v_T$$

is irreducible, we have

$$\mathbb{C}[S_i] s_k \cdot v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$$

where V^{λ_i} is the irreducible S_i-module indexed by $\lambda_i \in S_i^{\wedge}$.
Case II) Let $i < k$.
Then we have that s_k commutes with S_i by the Coxeter relations.

$$\mathbb{C}[S_i] \cdot s_k \cdot v_T = s_k \cdot \mathbb{C}[S_i] = s_k \cdot V^{\lambda_i}$$

Now to show: $s_k \cdot V^{\lambda_i} = V^{\lambda_i}$

The last being a result of the following homomorphism of irreducible representations

$$\phi : V^{\lambda_i} \to s_k V^{\lambda_i} \text{ where } v \mapsto s_k v$$

Coming from the representation preserving the action of the group.

$$\phi(gv) = g\phi(v) \; \forall g \in S_i$$

which is an isomorphism as $(\phi \circ \phi)(v) = s^2 v = v$

$$v_T \mapsto s_k v_T = \sum_{T'} a_{T'} v_{T'} \in s_k V^{\lambda_i} \hookrightarrow V^{\lambda_n} \text{where } T' = \lambda_0' \nearrow \ldots \nearrow \lambda_i'$$

Now we have that by case I) $\lambda_i = \lambda_k'$ for $i > k$ Case II) gives $\lambda_i = \lambda_k'$ for $i < k$ we conclude that $\lambda_i = \lambda_k'$ for $i \neq k$ □

Note: The explicit action of $s_k \cdot v_T$ will be given in section 4.
The action of the YJM-elements X_i on the Young basis is also local. It readily follows from

$$X_i = (\text{ Sum of all transpositions in } S_i) - (\text{ Sum of all transpositions in } S_{i-1})$$

that if

$$T = \lambda_0 \nearrow ... \nearrow \lambda_n$$

and we have the vector

$$\alpha(T) = (a_1, a_2, ..., a_n)$$

then a_k is the difference of a function of λ_k and a function of λ_{k-1} for all k.

Definition 1.2. Degenerate Affine Hecke Algebra $H(n)$
The unital associative algebra (over $\mathbb{C}$) generated by $Y_1, Y_2, ..Y_n$ and Coxeter generators $s_1, ..s_{n-1}$ with relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ where } |i-j| > 1,$$

$$Y_i Y_j = Y_j Y_i, \quad s_i Y_j = Y_j s_i \text{ for } j \neq i, i+1 \text{ and } s_i Y_i + 1 = Y_{i+1} s_i$$

Relations of $H(n)$ are stable under shifts of indices. $H(2)$ is the simplest example of a degenerate affine Hecke Algebra. It is generated by the elements Y_1, Y_2 and s. This algebra will play the central role in what follows. The irreducible finite-dimensional representations of $H(2)$ are either one-dimensional or two-dimensional. As Y_1 and Y_2 commute we can take v to be a common eigenvector and applying the involution s to v, we obtain a $H(2)$-invariant subspace of dimension at most 2. The importance of the algebra $H(2)$ is based on the following obvious yet useful fact.

Proposition 1.2. *The algebra $\mathbb{C}[S_n]$ is generated by the algebra $\mathbb{C}[S_{n-1}]$ and the algebra $H(2)$ with generators $Y_1 = X_{n-1}$, $Y_2 = X_n$, $s = s_{n-1}$, where X_{n-1} and X_n are the corresponding Young-Jucy-Murphy elements and $s_{n-1} = (n-1, n)$ is a Coxeter generator*

Of course, the algebra $\mathbb{C}[S_n]$ is generated by the subalgebra $\mathbb{C}[S_{n-1}]$ and one generator s_{n-1}, but it is taking into account the superfluous generators allows us to use induction. Each step from $\mathbb{C}[S_{n-1}]$ to $\mathbb{C}[S_n]$ reduces to the study of $H(2)$. Okounkov concludes this chapter with the following remark

Remark 1.1. *Consider the degenerate affine Hecke algebra $H(n)$. If we put $Y_1 = 0$, then the quotient of $H(n)$ modulo the corresponding ideal is canonically isomorphic to $\mathbb{C}[S_n]$.*

$$H(n)/(Y_1) \cong \mathbb{C}[S_n]$$

2 Irreducible Representations of $H(2)$

As previously mentioned all irreducible representations of $H(2)$ are at most two-dimensional and have an eigenvector v such that we can simultaneously diagonalize both elements.

$$Y_1 v = av, \quad Y_2 v = bv \quad a, b \in \mathbb{C}$$

Now if we suppose the vectors v and sv are linearly independent.
We can the following relation to deduce the actions $Y_1 sv$ and $Y_2 sv$.

$$Y_2 s = sY_1 + 1$$

We have

$$\begin{aligned}
(Y_2 s)v &= (sY_1 + 1)v \\
Y_2 sv &= sY_1 v + v \\
Y_2 sv &= sav + v \\
Y_2 sv &= v + asv
\end{aligned}$$

and similarly for Y_1

$$\begin{aligned}
sY_1 &= Y_2 s - 1 \\
Y_1 &= sY_2 s - s \\
Y_1 sv &= sY_2 ssv - ssv \\
Y_1 sv &= sY_2 v - v \\
Y_1 sv &= sbv - v \\
Y_1 sv &= -v + bsv
\end{aligned}$$

So that Y_1 and Y_2 act in the basis $\{v, sv\}$ as follows:

$$Y_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix} \; Y_2 = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \; s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we determine when we have a 1 dimensional sub-representation of the above.
Consider s acting on an eigenvector (c, d):

$$s \begin{pmatrix} c \\ d \end{pmatrix} = \gamma \begin{pmatrix} c \\ d \end{pmatrix} \text{ which can be re-written as } \begin{pmatrix} d \\ c \end{pmatrix} = \gamma \begin{pmatrix} c \\ d \end{pmatrix} \text{ where } \gamma = \pm 1$$

For $\gamma = 1$ we have $c = d$

$$Y_1 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - d \\ bd \end{pmatrix} = \begin{pmatrix} (a-1)d \\ bd \end{pmatrix} = \alpha \begin{pmatrix} c \\ d \end{pmatrix} \text{ So } \alpha = b \text{ and } \alpha = a - 1 \therefore b = a - 1$$

$$Y_2 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} bc + d \\ ad \end{pmatrix} = \begin{pmatrix} (b+1)d \\ ad \end{pmatrix} = \beta \begin{pmatrix} c \\ d \end{pmatrix} \text{ So } \beta = a \text{ and } \beta = b + 1 \therefore b = a - 1$$

Now when $\gamma = -1$ we have $c = -d$

$Y_1 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - d \\ bd \end{pmatrix} = \begin{pmatrix} (a+1)c \\ bd \end{pmatrix} = \alpha \begin{pmatrix} c \\ d \end{pmatrix}$ So $\alpha = b$ and $\alpha = a + 1 \therefore b = a + 1$

$Y_2 \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} bc + d \\ ad \end{pmatrix} = \begin{pmatrix} (b-1)c \\ ad \end{pmatrix} = \beta \begin{pmatrix} c \\ d \end{pmatrix}$ So $\beta = a$ and $\beta = b - 1 \therefore b = a + 1$

So if $b = a - 1$ then $v + sv$ spans a 1D subrepresentation and similarly if $b = a + 1$ then $v - sv$ spans a 1D subrepresentation.

Therefore if $b = a \pm 1$, then this 2D representation contains the unique one-dimensional subrepresentation given by

$$Y_1 \mapsto a, Y_2 \mapsto b = a \pm 1, \ s \mapsto \pm 1$$

in which v and sv are proportional.
Conversely, if we assume v and sv are proportional then we can show $b = a \pm 1$

$$\begin{aligned} sY_1 v + v &= Y_2 sv \\ sav + v &= bsv \\ (a-b)sv &= -v \\ sv &= \frac{-1}{a-b} v \text{ where } a \neq b \\ s^2 v &= \frac{1}{(a-b)^2} v = v \implies b = a \pm 1 \end{aligned}$$

If $a \neq b$ then the operators $\pi_{a,b}$ can be diagonalized as folllows:

$$Y_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ Y_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \ s = \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix} (*)$$

Note that always $a \neq b$ since otherwise the operators $\pi_{a,b}(Y_i)$ cannot be diagonalized and thus representations cannot occur in the action on the young basis. We now formulate our results as a proposition which describes representations in terms of transformations of weights (i.e eigenvectors).

Proposition 2.1. *Let $\alpha = (a_1, .., a_i, a_{i+1}, ..., a_n) \in Spec(n)$.*
Then $a_i \in \mathbb{Z}$ and

1. *$\forall i, \ \ a_i \neq a_{i+1}$ (Adjacent elements are not equal)*
2. *if $a_{i+1} = a_i \pm 1$, then $s_i \cdot v_\alpha = \pm v_\alpha$*
3. *if $a_{i+1} \neq a_i \pm 1$ then $\alpha' = s_i \cdot \alpha = (a_1, .., a_{i+1}, a_i, ...a_n) \in Spec(n)$ and $\alpha' \sim \alpha$ Moreover,*

$$v_{\alpha'} = (s_i - \frac{1}{a_{i+1} - a_i}) v_\alpha$$

and the elements s_i, X_i, X_{i+1} act in the basis $\{v_\alpha, v_{\alpha'}\}$ by $(*)$ with Y_1 replaced by X_i and Y_2 replaced by X_{i+1}.

In order to emphasize the role of Coxeter generators in the context of this section (as operators on weights α), we call them admissible transpositions.Admissible transpositions preserve $\mathrm{Spec}(n)$ and the set $\mathrm{Cont}(n)$.

Note that if $a_{i+1} \neq a_i \pm 1$ then in the basis

$$\{v_\alpha, c_i(s_i - d_i I)v_\alpha\} \text{ where } c_i = \frac{1}{a_{i+1} - a_i},\ d_i = \frac{1}{\sqrt{1 - c_i^2}}$$

the matrix of the transposition s_i is orthogonal:

$$s_i = \begin{pmatrix} \frac{1}{r} & \sqrt{1 - \frac{1}{r^2}} \\ \sqrt{1 - \frac{1}{r^2}} & \frac{-1}{r} \end{pmatrix} \text{ where } r = a_{i+1} - a_i$$

In Young's papers r is known as the **axial distance** and is the difference of the contents of the corresponding boxes in the young tableaux.

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