2 May 2021

The infinite-dimensional geometric story: Kac-Moody groups, affine flag varieties, and D-modules

Melbourne Representation Theory seminar, talk 2

Last week:
- motivation: no BB localisation in Kac-Moody setting, but still can examine D-modules
- Def'n of Kac-Moody groups, following Kumar
- Questions about different constructions of G → What's the point of Kumar's construction?
  Can we consider analytic manifold instead?

Today:
- flag varieties for general KM groups (still following Kumar)
- the affine case – KM groups as central extensions of loop groups
- aside: types of loop groups + an analogy
- the lattice model for SL_n
- example: SL_2

Next time (after Arun):
- When do we get equivalences w/ categories of D-modules?
- Sketch of proof of Kashiwara-Tanisaki negative level equivalence

↑ if there is still interest...
I'm happy to shut up if other people want to speak or if we're bored of this...

Recall our construction last week:

- input: $\mathfrak{g} = \mathfrak{g}(A)$ Kac-Moody Lie algebra
- construct: $\mathcal{U}$ = pro-unipotent pro-group with $\mathrm{Lie}\,\mathcal{U} = \hat{\mathfrak{n}} = \prod_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ completed nilpotent radical

  $\mathcal{U}_i$ = " " " $\mathrm{Lie}\,\mathcal{U}_i = \prod_{\Delta \setminus \{\alpha_i\}} \mathfrak{g}_\alpha$

  $T = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}^*_{\mathbb{Z}}, \mathbb{C}^*)$ torus (finite-dim'l)

  $B = \mathcal{U} \rtimes T$ $\quad N = \langle T, \{\tilde{s}_i\}\rangle$

  G_i = connected reductive alg. group w/ $\mathrm{Lie}\, G_i = \mathbb{C}f_i \oplus \mathfrak{h} \oplus \mathbb{C}e_i$

  $P_i = \mathcal{U}_i \rtimes G_i$ parabolic subgroups

$\rightsquigarrow$ The Kac-Moody group G of $\mathfrak{g}$ is the amalgamated product of $(N, P_i;\, i=1,\dots,\ell)$

nice properties:
- Fits into a Tits system (G, B, N, S) (Bruhat decomp, etc)
- $\{\text{pro-repns of } G\} \xleftrightarrow{1:1} \{\text{pro-reps of } \mathfrak{g} \text{ w/ integrable } \mathfrak{h}\text{-action}\}$

Flag Varieties

(I'll be brief/vague here so I can be more explicit in the affine case, which is what we really care about.)

- The homogeneous space $\mathcal{G}/\mathcal{B}$ can be given the structure of a projective ind-variety s.t. the Schubert varieties $X_w = \bigsqcup_{v \leq w} \mathcal{B} v \mathcal{B}/\mathcal{B}$ are closed projective subvarieties.

  A set X w/ filt'n $X_0 \subseteq X_1 \subseteq \cdots$
  - $\bigcup X_n = X$
  - X_n f.d. varieties
  - $X_n \hookrightarrow X_{n+1}$ closed

  all X_n are projective

- A sketch of this construction:
  - $\lambda \in \mathfrak{h}^*$ dominant integral, $V(\lambda)$ integrable highest wt $\mathfrak{g}$-module, $\mathcal{G} \curvearrowright V(\lambda)$ by previous theorem
    (e_i and f_i act locally nilpotently)
  - orbit through highest weight vector in $\mathbb{P}(V(\lambda))$ gives map $i_{V(\lambda)}: \mathcal{G}/\mathcal{B} \hookrightarrow \mathbb{P}(V(\lambda)) = \text{lines in } V(\lambda)$
    $g\mathcal{B} \longmapsto [g v_\lambda] = \text{line through } g v_\lambda$
  - $\mathcal{G}/\mathcal{B}$ has filt'n by $X_n = \bigcup_{\ell(v) \leq n} \mathcal{B} v \mathcal{B}/\mathcal{B}$
  - show $i_{V(\lambda)}(X_n)$ and $i_{V(\lambda)}(X_w)$ are closed in $\mathbb{P}(V(\lambda))$ for any $n \geq 0$

  $\Longrightarrow$ gives $\mathcal{G}/\mathcal{B}$ a (unique) projective ind-variety structure w/ filt'n $\{X_n\}_{n \geq 0}$ s.t. $i_{V(\lambda)}$ is closed embedding
  - a priori depends on λ, but can show that for $\mathfrak{g}$ symmetrizable, $\exists$ biregular iso $\mathcal{G}/\mathcal{B}(\lambda + \mu) \xrightarrow{\sim} \mathcal{G}/\mathcal{B}(\lambda)$ for any dominant integral $\mu \in \mathfrak{h}^*$

- The T-fixed points in $\mathcal{G}/\mathcal{B}$ are $\{\bar{w}\mathcal{B}\}_{w \in W} \in \mathcal{G}/\mathcal{B}$

- $\mathcal{U}^- \curvearrowright \mathcal{G}/\mathcal{B}$ and $\mathcal{U}^- \cdot id \subset \mathcal{G}/\mathcal{B}$ is open, inherits from $\mathcal{G}/\mathcal{B}$ ind-variety structure making it into an affine ind-group.

For $\mathfrak{g}$ affine, we can realise all of this more explicitly.

- Let $\mathring{\mathfrak{g}}$ = f.d. simple Lie algebra, $\mathfrak{g}$ = corresponding affine Lie algebra (central extension of loop algebra $\mathring{\mathfrak{g}}[t^{\pm 1}]$)

  $\mathring{G}$ = connected, simply conn'd alg. gp w/ Lie $\mathring{G} = \mathring{\mathfrak{g}}$ $\quad$ G = Kac-Moody group w/ Lie $G = \mathfrak{g}$

<u>defn</u> For R = associative $\mathbb{C}$-algebra, $\mathring{G}(R)$ = set of R-rational points of $\mathring{G}$; i.e. set of $\mathbb{C}$-algebra homs $\mathbb{C}[\mathring{G}] \longrightarrow R$. $\mathring{G}(R)$ has canonical group structure.

- e.g. if $\mathring{G} = SL_n(\mathbb{C})$, $\mathring{G}(R) = SL_n(R)$

- For $R = \mathbb{C}((t))$ Laurent series (i.e. finitely many powers of t^{-1}, infinitely many powers of t), this yields the loop group of $\mathring{G}$:

$$\boxed{\mathcal{L}(\mathring{G}) := \mathring{G}(\mathbb{C}((t)))}$$

• Extend $\mathcal{L}(\mathring{G})$ by "exp d": should be "exp c" ?!

- consider group hom $\gamma : \mathbb{C}^\times \longrightarrow \mathrm{Aut}(\mathbb{C}((t)))$, $\gamma(z)(p(t)) = p(zt)$ "loop rotation"

- get $\gamma_{\mathring{G}} : \mathbb{C}^\times \longrightarrow \mathrm{Aut}\,\mathcal{L}(\mathring{G})$

- Define $\overline{\mathcal{L}}(\mathring{G}) := \mathbb{C}^\times \ltimes \mathcal{L}(\mathring{G})$

• Relationship between $\overline{\mathcal{L}}(\mathring{G})$ and $\mathcal{G}$:

center of $\mathring{G}$

<u>Theorem:</u> $\exists$ group hom $\Psi : \mathcal{G} \longrightarrow \overline{\mathcal{L}}(\mathring{G})/C$ s.t. Ψ is surjective and $\ker\Psi =$ center of $\mathcal{G}$,

$\Rightarrow$ We can study $\overline{\mathcal{L}}(\mathring{G})$ instead of $\mathcal{G}$ and we basically lose no information

• subgroups of $\overline{\mathcal{L}}(\mathring{G})$:

$$
\begin{tikzcd}
 & \overline{\mathcal{L}}(\mathring{G}) & \\
 & \mathbb{C}^\times \ltimes \mathring{G}(\mathbb{C}[[t]]) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "ev_0"] & \mathbb{C}^\times \times \mathring{G} \\
\text{Standard Iwahori} & I := ev_0^{-1}(\mathbb{C}^\times \times \mathring{B}) \arrow[u, phantom, sloped, "\subset"] \arrow[r] & \mathbb{C}^\times \times \mathring{B} \arrow[u, phantom, sloped, "\subset"] \\
\text{unipotent radical} \quad \mathcal{U} \arrow[r, "\text{iso if } c=0", "\sim"' above, "\Psi"'] & I_u := ev_0^{-1}(\mathbb{C}^\times \times \mathring{N}) \arrow[u, phantom, sloped, "\subset"] \arrow[r] & \mathbb{C}^\times \times \mathring{N} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

pro-unipotent pro-group defined earlier w/ $\mathrm{Lie}\,\mathcal{U} = \hat{\mathfrak{n}}$ (completion)

$\mathring{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathring{\mathfrak{h}}^*, \mathbb{C}^\times) \subset \mathring{G}$ max'l torus

$T := \mathbb{C}^\times \times \mathring{T} \subset \overline{\mathcal{L}}(\mathring{G})$

Standard max'l torus

<u>Thm:</u> Ψ induces bijection

$$\mathcal{G}/\mathcal{B} \xrightarrow{\sim} \overline{\mathcal{L}}(\mathring{G})/I$$

A quick remark on central extensions:

Set-up w/ central extension:

$$\begin{tikzcd}
\overline{\mathcal{L}}(\mathring{G}) & \\
\cup & \\
\mathbb{C}^\times \ltimes \mathring{G}(\mathbb{C}[[t]]) \arrow[r, "ev_0"] & \mathbb{C}^\times \times \mathring{G} \\
\cup & \cup \\
\overline{I} := ev_0^{-1}(\mathbb{C}^\times \times \mathring{B}) \arrow[r] & \mathbb{C}^\times \times \mathring{B} \\
\cup & \cup \\
\overline{I}_u := ev_0^{-1}(\mathbb{C}^\times \times \mathring{N}) \arrow[r] & \mathbb{C}^\times \times \mathring{N}
\end{tikzcd}$$

all contain $\mathbb{C}^\times$

set-up w/o central extension.

$$\begin{tikzcd}
\mathcal{L}(\mathring{G}) = \mathring{G}((t)) & \\
\cup & \\
\mathring{G}[[t]] \arrow[r, "ev_0"] & \mathring{G} \\
\cup & \cup \\
I \arrow[r] & \mathring{B} \\
\cup & \cup \\
I_u \arrow[r] & \mathring{N}
\end{tikzcd}$$

• Flag varieties don't see the difference:

"the affine flag variety"

$$\boxed{\overline{\mathcal{L}}(\mathring{G}) / \overline{I} = \mathring{G}((t)) / I}$$

"the affine grassmannian"

$$\boxed{\overline{\mathcal{L}}(\mathring{G}) / \mathbb{C}^\times \ltimes \mathring{G}[[t]] = \mathring{G}((t)) / \mathring{G}[[t]]}$$

• Also, as Kari pointed out in his email, flag varieties also don't see the completion:

$$\boxed{\overline{\mathcal{L}}(\mathring{G}) / \overline{I} = \mathring{G}((t)) / I = \mathring{G}[t^{\pm 1}] / I \cap \mathring{G}[t]}$$

What's the point of introducing the central extension? Or completing?

• $\overline{\mathcal{L}}(\mathring{G})$, $\mathring{G}((t))$, and $\mathring{G}[t^{\pm 1}]$ act on these homogeneous spaces. There are certain line bundles which are not $\mathring{G}((t))$-equivariant, but <u>are</u> $\overline{\mathcal{L}}(\mathring{G})$-equivariant. So it's good to remember the $\overline{\mathcal{L}}(\mathring{G})$-action.

But I don't understand this very well so I can't really tell you more...

Aside: types of loop groups

We called $\mathring{G}((t))$ a "loop group." What does this have to do with loops? Aren't there "more loopy" definitions?

reductive alg. gp/$\mathbb{C}$

K compact Lie group

- What are "loops in K"? Well, they're

$$\mathcal{L}K = \mathrm{Maps}(S^1, K)$$

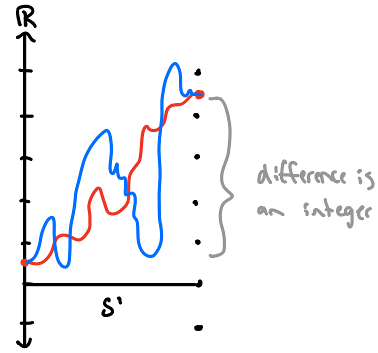
- But what kind of maps?
  - continuous? ⟵ As Kari pointed out, perhaps this is the most natural.
  - differentiable? ⟵ These showed up originally in physics as "gauge groups/current groups", corresponding loop groups have significance in 2D quantum field theory
  - real analytic?
  - polynomial? Will explain what this means in a moment...

"In fact, it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups" Segal, Loop groups

• Let's think in the simplest example, $K = S^1$

$$\mathrm{Maps}(S^1, S^1) = \mathrm{Maps}(S^1, \mathbb{R}/\mathbb{Z})$$

$\Rightarrow$ There are tons of maps!



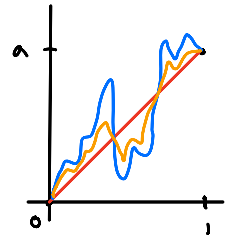
• For $a \in \mathbb{Z}$, consider $\mathrm{Maps}_{0,a} := \{ f : [0,1] \longrightarrow \mathbb{R} \mid f(0) = 0,\ f(1) = a \}$

$\Big\downarrow$ contracts to a single map

corresponds to

$z \in S^1 \longmapsto z^a$ $\longrightarrow$ $f_a : [0,1] \longrightarrow \mathbb{R}, \quad x \longmapsto ax$

"polynomial loop"



• So if we look at the space of basepoint preserving maps (smooth, continuous, whatever), it contracts the space of polynomial loops:

$$\mathrm{Maps}(S^1, S^1)_{id \to id} = \bigsqcup_{a \in \mathbb{Z}} \mathrm{Maps}_{0,a} \underset{\text{homotopic}}{\sim} \underbrace{\{ f_a,\ a \in \mathbb{Z} \}}_{\text{polynomial loops}} \underset{\text{isomorphic}}{\sim} \mathbb{Z}$$

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- This is a good approximation the general picture:

$$\mathrm{Maps}(S^1,K)_{id \to id} \underset{\text{homotopic}}{\sim} \Omega_{poly}(S^1,K)_{id \to id}$$

(
- An analogy:

$\underset{\text{real Lie group}}{G_{\mathbb{R}}} \supset \underset{\text{max'l compact}}{K_{\mathbb{R}}} \rightsquigarrow G_{\mathbb{R}}/K_{\mathbb{R}}$ contractible

- To understand $G_{\mathbb{R}}$, we better understand $K_{\mathbb{R}}$ first

in our setting, $\mathcal{L}_{cts}K / \Omega_{poly}K$ is contractible

- So maybe it's good to study Ω first
)

- What are polynomial loops in general?

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, phantom, "\cap"] & K \arrow[d, phantom, "\cap"] \\ \mathbb{C}^\times \arrow[r, "\tilde{f}"] & G \end{tikzcd}$$

complexification of K

f is polynomial if it is the restriction of an algebraic map $\tilde{f} : \mathbb{C}^\times \to G$

- We saw this for $K = S^1$. What do polynomial loops look like in general?

- A bigger example: $K = SU_2$

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, phantom, "\cap" rotate=-90] & SU_2 \arrow[d, phantom, "\cap" rotate=-90] \\ \mathbb{C}^\times \arrow[r, "\tilde{f}"] & SL_2(\mathbb{C}) \end{tikzcd}$$

$\tilde{f} : \mathbb{C}^\times \longrightarrow SL_2(\mathbb{C})$ algebraic map s.t. $\tilde{f}(S^1) \subset SU_2$

$\downarrow$

$\tilde{f} : a \longmapsto \begin{pmatrix} f_1(a) & f_2(a) \\ f_3(a) & f_4(a) \end{pmatrix}$ for some $f_i \in \mathbb{C}[t^{\pm 1}]$

$\Rightarrow \tilde{f} \in SL_2(\mathbb{C}[t^{\pm 1}])$

- The condition $\tilde{f}(S^1) \subset SU_2$ amounts to a set of equations (maybe involving complex conjugation)

$\rightsquigarrow \Omega_{poly}(S^1, K)$ has the structure of a real algebraic ind-variety

- Because $\Omega_{poly}(S^1, K) \subset G(\mathbb{C}[t^{\pm 1}])$, get map

$$\underbrace{\Omega_{poly}(S^1, K)}_{\text{real ind-variety}} \longrightarrow \overbrace{\underbrace{G(\mathbb{C}[t^{\pm 1}]) / G(\mathbb{C}[t])}_{\text{complex ind-variety}}}^{\text{affine Grassmannian}}$$

Fact: This map is a homeomorphism $\Omega_{poly}(S^1, G) \simeq Gr$ (see Zhu's notes on affine Grassmannians)

Moral: "Something that looks real is actually complex"

- Let's take a step back and take stock of the big picture. We'll start with an analogy:

- A familiar story: K compact Lie group, G complexification

$$\begin{tikzcd}
K \arrow[d, bend left, "\text{complexify}"] & \mathfrak{g}_{\mathbb{R}} \arrow[l, "\text{local diffeo}"'] \arrow[d, bend left, "\otimes_{\mathbb{R}} \mathbb{C}"] \\
G \arrow[u, bend left, "\text{max'l cpt}"] & \mathfrak{g}_{\mathbb{C}} \arrow[l, "\text{local diffeo}"] \arrow[u, bend left, "\text{cpt real form}"]
\end{tikzcd}$$

Study rep'ns using geometry

manifolds		algebraic varieties	
K/T	$\sim$	G/B	holomorphic line bundles provide all irreps of K
K/L (Levi) (e.g. $SU(2) \subset SU(3)$)	$\sim$	G/P_L	holomorphic line bundles provide some irreps of K (but space is easier)

ex: $SU_n / SU_{n-1} \cong SL_n(\mathbb{C}) \Big/ \left(\begin{array}{c|c} * & * \\ \vdots & \\ 0 & * \end{array}\right) \cong \mathbb{P}^{n-1}(\mathbb{C})$

$\mathcal{O}(1)$ provides natural rep'n

<u>Claim:</u> (Milnor, Remarks on infinite-dim'l Lie groups, 1984)

We can't expect a good dictionary between infinite-dim'l Lie groups and infinite-dim'l Lie algebras in general

- e.g. G = diffeomorphisms of the circle is bad (according to Segal, I don't really understand why...)

Amazing thing about loop groups: all of the above generalises

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The loop group picture: (extracted from Segal "loop groups" - the spirit is this but I may have a few details wrong...)

$$\begin{tikzcd} \Omega_{poly}(S^1, K) \arrow[d, shift left] & \mathfrak{g}_{\mathbb{R}}[t^{\pm 1}] \arrow[l] \arrow[d, shift left] \\ G((t)) \arrow[u, shift left] & \mathfrak{g}_{\mathbb{C}}[t^{\pm 1}] \arrow[l] \arrow[u, shift left] \end{tikzcd}$$

"loops into complexification"

∞-dim'l manifolds		ind-varieties	
$\Omega_{poly}(S^1,K)/T_K$	$\sim$	$G((t))/I$ affine flag variety	holomorphic line bundles provide all integrable reps
$\Omega_{poly}(S^1,K)/K$	$\sim$	$G((t))/G[[t]]$ affine Grassmannian	" " some reps

- **ex:** $\Gamma(Gr, \mathcal{O}(1)) =$ "basic rep'n" $= L(\Lambda_0)$

What about rep'n theory?

in our analogy:

$$\begin{array}{ccc} \text{unitary rep'ns of } K & \sim & \text{f.d. rep'ns of } \mathfrak{g}_{\mathbb{R}}/\mathbb{C} \\ \wr\!\wr & & \wr\!\wr \\ \text{rep'ns of } G & \sim & \text{f.d. rep'ns of } \mathfrak{g}_{\mathbb{C}} \end{array}$$

$\rightsquigarrow$ The <u>same holds</u> in the loop group picture above (replace "f.d." with "integrable") (according to Segal...)

The Upshot: The natural thing to study is the loop group $LK = \mathrm{Maps}_{smooth}(S^1, K)$ as Kari suggested in his email. But we can study the rep'n theory of LK by studying the rep'n theory of the loop algebra $\mathfrak{g}_{\mathbb{C}}[t^{\pm 1}]$, which is easier (in the same way semisimple Lie algebras are easy), and its rep theory can be described in terms of the complex ind-varieties $\mathcal{F}\ell$ and Gr.

One last comment on this picture: central extensions

In analogy: What if K isn't simply connected? $\longrightarrow$ Take its universal cover, a central extension

e.g. $K = SO(3)$ e.g. $SU(2) \twoheadrightarrow SO(3)$

Then life is easier.

In loop groups: To make life easier, we also take a central extension:

$$\begin{tikzcd}
\widehat{\Omega}_{poly}(S^1,K) \arrow[r, two heads] & \Omega_{poly}(S^1,K) \arrow[d, shift left] & \mathfrak{g}_{\mathbb{R}}[t^{\pm1}] \arrow[l] \arrow[d, shift left] & \hat{\mathfrak{g}}_{\mathbb{R}}[t^{\pm1}] \arrow[l, two heads] \\
\widehat{G((t))} \arrow[r, two heads] & G((t)) \arrow[u, shift left] & \mathfrak{g}_{\mathbb{C}}[t^{\pm1}] \arrow[l] \arrow[u, shift left] & \hat{\mathfrak{g}}_{\mathbb{C}}[t^{\pm1}] \arrow[l, two heads]
\end{tikzcd}$$

Kac-Moody group!

affine Lie algebra!

- The central extension is not seen by the affine flag varieties, so the geometry doesn't change.

Lattice model for SL_n:

defn) A lattice in $\mathbb{C}((t))^n$ is a $\mathbb{C}[\![t]\!]$-submodule $\mathcal{L} \subset \mathbb{C}((t))^n$ s.t. $\mathbb{C}((t)) \otimes_{\mathbb{C}[\![t]\!]} \mathcal{L} \simeq \mathbb{C}((t))^n$

- the standard lattice is $\mathcal{L}_0 = \mathbb{C}[\![t]\!]^n$

- Have natural action $GL_n((t)) \curvearrowright \{\text{lattices in } \mathbb{C}((t))^n\}$ $\Rightarrow$ $\underset{\text{affine Grassmannian}}{\overset{Gr}{\shortparallel}}$ $GL_n((t)) / GL_n[\![t]\!] \simeq \left\{ \begin{matrix} \text{lattices} \\ \text{in} \\ \mathbb{C}((t))^n \end{matrix} \right\}$ set-theoretically
  - transitive.
  - $\mathrm{Stab}_{GL_n((t))} \mathcal{L}_0 = GL_n[\![t]\!]$

- We can put an ind-scheme structure on $\{\text{lattices}\}$ as follows:
  - Set $Gr^N = \{ \mathcal{L} \mid t^{-N}\mathcal{L}_0 \supset \mathcal{L} \supset t^N \mathcal{L}_0 \}$ lattices whose poles and zeroes have order $\leq N$
  - Note that:

$$
\begin{aligned}
Gr^N &\overset{1:1}{\longleftrightarrow} \{ t\text{-stable subspaces of } t^{-N}\mathcal{L}_0 / t^N \mathcal{L}_0 \} \\
\mathcal{L} &\longmapsto \text{image of } \mathcal{L} \text{ in } t^{-N}\mathcal{L}_0 / t^N \mathcal{L}_0 \\
\varphi^{-1}(M) &\longleftarrow\!\shortmid\; M
\end{aligned}
$$

here $\varphi : t^{-N}\mathcal{L}_0 \to t^{-N}\mathcal{L}_0 / t^N \mathcal{L}_0$

- What does $t^{-N}\mathcal{L}_0 / t^N\mathcal{L}_0$ look like?

  - Well, as a vector space it is 2N copies of $\mathbb{C}^n$, with a t-action given by:

$$\underset{\text{coeffs of } t^{-N}}{\mathbb{C}^n} \overset{t}{\to} \oplus \underset{\text{coeffs of } t^{-N+1}}{\mathbb{C}^n} \overset{t}{\to} \oplus \cdots \overset{t}{\to} \oplus \underset{\text{coeffs of } t^{N}}{\mathbb{C}^n} \simeq \mathbb{C}^{2nN}$$

  - The subspace corresponding to $\mathcal{L}_0$ is $0 \oplus 0 \oplus \cdots \oplus 0 \oplus \overbrace{\underset{\text{coeffs of } 1}{\mathbb{C}^n} \oplus \cdots \oplus \underset{\text{coeffs of } t^N}{\mathbb{C}^n}}^{N}$ (a nN-dim'l subspace)

$\Rightarrow$ get an embedding

$$Gr^N \hookrightarrow \mathrm{Grass}(\mathbb{C}^{2nN}) = \bigsqcup_{d \geq 0} \mathrm{Grass}(\mathbb{C}^{2nN}, d)$$

Grassmannian of subspaces of $\mathbb{C}^{2nN}$

  - give Gr^N the necessary topology to make this embedding closed.

    $\Rightarrow Gr^N$ gains the structure of a finite-dim'l projective variety

So $Gr = \bigcup_{N \geq 0} Gr^N$ gains the structure of a projective ind-variety.

<u>Theorem:</u> This agrees w/ the ind-variety structure I sketched earlier

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For SL_2 we can be even more explicit:

$$Gr_{SL_2} = \{\text{lattices in } \mathbb{C}((t))^2 \text{ of det } 1\}$$

right action by $SL_2[[t]]$ lets us do column operations, so can reduce to this form

- represent a lattice $\mathcal{L}$ by a matrix

$a,b,c,d \in \mathbb{C}((t))$

$\mathcal{L}$ = span of columns

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

min val occurs in 2nd row or both rows:

$$\left\{ \begin{pmatrix} t^m & f \\ 0 & t^{-m} \end{pmatrix} : f = f_{-m}t^{-m} + \cdots + f_{m-1}t^{m-1} \right\} \cong \mathbb{C}^{2m} \quad \text{cell } C^{2m}$$

min val occurs in 1st row:

$$\left\{ \begin{pmatrix} t^{-m} & 0 \\ g & t^{m} \end{pmatrix} : g = g_{-m+1}t^{-m+1} + \cdots + g_{m-1}t^{m-1} \right\} \cong \mathbb{C}^{2m-1} \quad \text{cell } C^{2m-1}$$

- let $-m$ = minimal valuation ($m \geq 0$)
(i.e. m = max order of pole in a,b,c,d)

Get cell decomposition

$$Gr_{SL_2} = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^3 \sqcup \cdots \quad \text{and} \quad \overline{C^i} = \bigsqcup_{j \leq i} C^j$$

Schubert cells of affine Grassmannian
(parametrized by cocharacter lattice)

Sadly I've run out of time, but can also do this for any group $\mathring{G} \hookrightarrow SL_n$, and can consider $\{(\text{lattice}, \overset{\text{line}}{\ell} \subset \mathcal{L})\}$ as a model for $\mathcal{F}\ell_{SL_2} = SL_2((t))/I$. More in two weeks!