D-modules on ind-schemes and representations of affine Kac-Moody algebras

Melbourne Representation theory Seminar - talk 3

18 May, 2021

Previous talks:
- We constructed a group, the Kac-Moody group, associated to a Kac-Moody algebra.
- For affine Kac-Moody algebras, it was an extension of the loop group $G((t))$.
- We discussed various types of loop groups (smooth, analytic, polynomial) and described why they lead us to study $G((t))$.
- We constructed ind-varieties $\mathcal{G}/P$ associated to a Kac-Moody group $\mathcal{G}$, and described (/stated) how we can get $\mathcal{G}$-rep'ns / integrable $\mathfrak{g}$-modules as global sections of line bundles ("Borel-Weil") on $\mathcal{G}/P$'s.
- In the affine case, this led us to the affine flag variety $\mathcal{F}\ell = G((t))/I$ and the affine Grassmannian $Gr = G((t))/G[\![t]\!]$.

The point of doing all this: study $\mathfrak{g}$-modules using D-modules on $\mathcal{F}\ell$ and Gr

Today we finally do this.

TODAY'S GOALS:
1. explain (roughly) what D-modules on ind-schemes are
2. sketch proof of negative level equivalence $D(\mathcal{F}\ell)^{\lambda\ldots}\text{-mod} \xrightarrow{\sim} \mathcal{O}_{\nu(\lambda)}$

Warning/Remark: I am very much NOT an expert on this material and I'm just trying to make heads or tails of a vast and very technical literature on this subject. There are still many things I don't understand, but at least I'm trying!

My approach: Try to get a birds eye view of what's going on instead of nailing down all of the technical details.

Recall that we don't expect to get a BB-type equivalence of categories for affine $\mathfrak{g}$:

$$\underset{\text{affine KM group}}{\widehat{G}((t))} \curvearrowright \mathcal{F}\ell \Rightarrow \underset{\text{affine KM algebra}}{\mathfrak{g}} \longrightarrow \Gamma(\mathcal{F}\ell, D(\mathcal{F}\ell))$$

we don't know what this is yet, but whatever it is, it should have this property...

one may hope: $\mathfrak{g}$ generates a dense subalgebra of $\Gamma(\mathcal{F}\ell, D(\mathcal{F}\ell))$

(i.e. $\Gamma(\mathcal{F}\ell, D(\mathcal{F}\ell))$ is a completion of $U(\mathfrak{g})$ by a certain topology)

- True in f.d. setting b/c of "Kostant's normality theorem"

$$\Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \simeq S(\mathfrak{g}) / (S(\mathfrak{g})^G_+)$$

- False in affine setting ([Beilinson-Drinfeld, Remark 7.15.7 (ii)])

$$\mathfrak{g} \longrightarrow \Gamma(\mathcal{F}\ell, D(\mathcal{F}\ell)) \quad \text{is not surjective.}$$

<u>Remark:</u> Arun suggested two weeks ago that we could just "find a different ideal" when I claimed that $\mathfrak{g}/(c\,\mathbb{C}[c]) \longrightarrow \Gamma(\mathcal{F}\ell, D(\mathcal{F}\ell))$ wasn't an equivalence. I liked this idea a lot, but the fact above shows that it won't work

So no equivalence, but we still have functor $\Gamma: D(\mathcal{F}\ell)\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$

What is a D-module on $\mathcal{F}\ell$?

Two approaches:

① (Kashiwara-Tanisaki) We've described $\mathcal{F}\ell$ very explicitly as an ind-scheme:

$$\mathcal{F}\ell = \bigsqcup_{w \in W} X_w \qquad \underbrace{X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = \bigsqcup_{\ell(w) \le n} X_w \hookrightarrow \cdots}_{\text{closed embeddings}} \qquad \mathcal{F}\ell = \bigcup_{n \ge 0} X_n$$

Use this to define

$$D(\mathcal{F}\ell)\text{-modules} = \lim_{n \to \infty} D(X_n)\text{-modules}$$

X_n is singular, but we can embed it in a smooth f.d. variety as a closed subvariety, then make sense of $D(X_n)$-modules using Kashiwara's theorem. Then we can realise $D(X_n)\text{-mod} \overset{\text{full}}{\underset{\text{subcat}}{\subseteq}} D(X_{n+1})\text{-mod}$

This is done carefully in [Kashiwara-Tanisaki Negative level I and II].

- can add λ-twists for $\lambda \in \mathfrak{h}^*$ (which include level, as $c \in \mathfrak{h}$)

PROS:
- reduces to f.d. settings
- can show $\Gamma : D^\lambda(\mathcal{F}\ell)\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$ is exact using same argument as f.d. setting (induction on n) when λ is s.t. $M(\lambda)$ is irreducible ($\Leftarrow \lambda$ is negative level)
- can show $\Gamma(i_{w!}\mathcal{O}_{X_w}) = \text{Verma}$, $\Gamma(i_{w*}\mathcal{O}_{X_w}) = \text{dual Verma}$ $\longrightarrow$ enough to establish KL conjecture for negative level λ

CONS:
- quite specific to $\mathcal{F}\ell$, which is very nice (formally smooth ind-scheme of ind-finite type)
- not clear that Γ is exact away from negative level

② (Beilinson-Drinfeld) Build a D-module theory that works on <u>all</u> ind-schemes ("D-crystals")
(of ind-finite type)

- Specialize to formally smooth ind-schemes

For every k-algebra A and nilpotent ideal $I \subset A$, $X(A) \twoheadrightarrow X(A/I)$ is surjective (note: if X is finite-type, smooth $\Leftrightarrow$ formally smooth)

Rough sketch of how this works:

- two notions of $\mathcal{O}$-modules on ind-scheme $X = \varinjlim X_\alpha$ $\qquad i_{\alpha\beta}: X_\alpha \hookrightarrow X_\beta$

the more obvious def'n: $\mathcal{O}^p$-modules: "pro" — collection of $\mathcal{O}$-modules on X_α's: $\{P_\alpha\}$ w/ identifications $i_{\alpha\beta}^* P_\beta = P_\alpha$

the more useful def'n: $\mathcal{O}^!$-modules: "ind" — collection of $\mathcal{O}$-modules on X_α's: $\{M_\alpha\}$ w/ identifications $i_{\alpha\beta}^! M_\beta = M_\alpha$

sections supported on X_α: $i_{\alpha\beta}^!(M_\beta) := Hom_{\mathcal{O}_{X_\beta}}(\mathcal{O}_{X_\alpha}, M_\beta)$

- Use $\mathcal{O}^!$-modules to define differential operators:
  - <u>Step 1</u>: For $M \in \mathcal{O}^!$-mod, define differential morphisms

$$Der(\mathcal{O}_X, M) := \varinjlim Der(\mathcal{O}_{X_\alpha}, M_\alpha)$$

$$D(M) = Diff(\mathcal{O}_X, M) := \varinjlim Diff(\mathcal{O}_{X_\alpha}, M_\alpha)$$

  - for a given X_α, $Diff(\mathcal{O}_{X_\alpha}, M_{X_\alpha})$ is both left and right $\mathcal{O}_{X_\alpha}$-module

$\rightsquigarrow$ can think of it as $\mathcal{O}_{X_\alpha \times X_\alpha}$-module supported set-theoretically on diagonal

$\rightsquigarrow D(M)$ is an $\mathcal{O}^!$-module on $X \times X$ supported set-theoretically on diagonal

<u>Step 2</u>: generalise this to define Diff-bimodules on X

$Y \subset X$ "reasonable subscheme" $\longmapsto$ $D_Y =$ $\mathcal{O}^!$-module on $Y \times X$ supported set-theoretically on diagonal $Y \subset Y \times X$ $+$ for $Y \subset Y'$, $D_{Y'} \otimes \mathcal{O}_Y \xrightarrow{\sim} D_Y$ identifications

closed q.compact subscheme s.t. for any closed $Z \subset Y$, the ideal of Y in $\mathcal{O}_Z$ is f.g.

- the category $\mathcal{M}^{di}(X, \mathcal{O})$ of these is a monoidal $\mathbb{C}$-category: $(D \otimes D')_Y := \varinjlim (D_Y)_{Y \times Y'} \otimes_{\mathcal{O}_{Y'}} D'_{Y'}$
- A diff-algebra on X is an algebra object in $\mathcal{M}^{di}(X, \mathcal{O})$
- A D-module on X (for a chosen diff-algebra D) is a (necessarily right) D-module in $\mathcal{M}^{di}(X, \mathcal{O})$
- Using this, can define our favourite diff-algebra D_X ("sheaf of diff'l operators")

$$Y \longmapsto (D_X)_Y = \varinjlim \mathrm{Diff}(\mathcal{O}_{Y'}, \mathcal{O}_Y)$$

- <u>Fact</u>: $D_X\text{-mod} = \varinjlim_{\alpha} D_{X_\alpha}\text{-mod}$ so this construction agrees w/ the previous one

PROS:
- more general (not just $\mathfrak{Fl}$)
- intrinsic def'n (doesn't depend on X_α's)

CONS:
- if we specialize to $\mathfrak{Fl}$, its still not clear that Γ is exact away from negative level

A third strategy: Define D-modules on $G((t))$

Why would this be useful? A motivational proof: $\mathfrak{g}$ simple Lie alg $/\mathbb{C}$, G alg group adjoint type $\supset B$

Theorem (BB): $\Gamma: D^\lambda(G/B) \longrightarrow \mathfrak{g}\text{-mod}$ is exact.

proof (sketch) (Frenkel-Gaitsgory)

- Projection $\pi: G \longrightarrow G/B$ gives functor $D^\lambda(G/B)\text{-mod} \xrightarrow{\pi^*} D(G)\text{-mod}$
- For any $\mathcal{F}' \in D(G)\text{-mod}$, $\Gamma(G, \mathcal{F}')$ is a $\mathfrak{g}$-bimodule
- For $\mathcal{F} \in D^\lambda(G/B)\text{-mod}$, we have

projection formula + Frobenius reciprocity

$$\Gamma(G/B, \mathcal{F}) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}^{-\lambda}, \Gamma(G, \pi^*(\mathcal{F})))$$

$\mathfrak{b}$-module via right action of $\mathfrak{b}$.

- As a right $\mathfrak{g}$-module, $\Gamma(G, \pi^*(\mathcal{F}))$ is in category $\mathcal{O}$ (b/c $\pi^*\mathcal{F} \in$ (right) B-eqvt $D(G)$-mod $\xrightarrow{\Gamma}$ $(\mathfrak{g}, B)$-mod $=$ $(\mathfrak{g}, \mathfrak{g})$-mod where right $\mathfrak{b}$-action is integrable) — left, right

$\Rightarrow$ get a functor $\Gamma': D^\lambda(G/B)\text{-mod} \longrightarrow \mathcal{O}$, $\mathcal{F} \longmapsto \Gamma(G, \pi^*(\mathcal{F}))$

- exact b/c π is a submersion (smooth $\Rightarrow$ flat) and G is affine

- Then can realise global sections as a composition

$$\Gamma(G/B, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}}(M(-\lambda), \underbrace{\Gamma'(\mathcal{F})}_{\text{exact}})$$

- $\mathrm{Hom}_{\mathcal{O}}(M(-\lambda), -)$ exact when $M(-\lambda)$ projective in $\mathcal{O}$
- $M(\mu)$ projective $\Leftrightarrow$ $\mu+\rho$ dominant

$\square$

Claim (Frenkel-Gaitsgory): If we can make sense of D-modules on $G((t))$, the argument above works basically word-for-word in the affine setting

(level b/c κ = multiple of Killing form, critical level = $\kappa = -\frac{1}{2}$ Killing form)
FG set-up: $\mathfrak{g}$ simple Lie alg / $\mathbb{C}$, κ invariant inner product on $\mathfrak{g}$, $0 \to \mathbb{C}1 \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$

Key fact: The vacuum Weyl module $\mathbb{V}_{\mathfrak{g},\kappa} = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}1}^{\hat{\mathfrak{g}}_\kappa}(\mathbb{C})$ is projective in the appropriate category $\mathcal{O}$ if κ is positive or irrational.

Critical level: $\mathbb{V}_{\mathfrak{g},crit}$ is not projective in $\mathfrak{g}_{crit}$-mod, so the argument above doesn't hold word-for-word. BUT, it can be modified:

- $\mathbb{V}_{\mathfrak{g},crit}$ projective in integrable $\mathfrak{g}_{crit}$-modules supported on $\mathrm{spec}(\mathfrak{z}_{\mathfrak{g}})$ center of completed enveloping algebra, opers etc
- Can do some work to use this to show $\Gamma: D^{crit}(Gr)\text{-mod} \to \mathfrak{g}_{crit}\text{-mod}$ is exact

The upshot: Our definitions of D-modules on ind-schemes are insufficient for studying critical level rep'ns. To fix it, we need a notion of D-modules on the loop group $G((t))$.

Why is this hard?

Because $G((t))$ is both "pro" and "ind"

$$\cdots \quad t^{-3} \quad t^{-2} \quad t^{-1} \quad t^{0} \quad t^{1} \quad t^{2} \quad t^{3} \quad \cdots$$

$\xleftarrow{\text{ind}}$ (only finitely many powers of t^{-1})

$\xrightarrow{\text{pro}}$ (infinitely many powers of t)

- ex: Consider $\mathbb{C}((t))$. How do we build this in algebraic geometry?

first: $\mathbb{C}[[t]]$, as a scheme this is just $\underset{\text{coeff of } t^0}{\mathbb{C}} \times \underset{t^1}{\mathbb{C}} \times \underset{t^2}{\mathbb{C}} \times \cdots$

$$= \operatorname{Spec} \mathbb{C}[x_0, x_1, x_2, \dots] = \lim_{n\to\infty} \operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]$$

ind-object in rings
= pro-object in schemes

problem: not finite type

$\rightsquigarrow$ either get scheme of infinite type or pro-scheme of pro-finite type

Then: $\mathbb{C}((t)) = \bigcup_{i\geq 0} t^{-i}\, \mathbb{C}[[t]]$ an ind-object in whatever world $\mathbb{C}[[t]]$ lives in

Moral: $\mathbb{C}((t))$ is an ind-pro-scheme of finite type. So we can't use our previous constructions of D-modules on ind-schemes.

So how do we define D-modules on $G((t))$?

Arkhipov - Gaitsgory:

$$D\text{-modules on } G((t)) = \text{modules over a chiral algebra } \mathbb{V}_{G,\kappa} = \mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\mathfrak{g}[[t]]\oplus\mathbb{C}1}\left(\mathcal{O}_{G[[t]]}\right)$$

- $\mathbb{V}_{G,\kappa}$ is a vertex operator algebra (= chiral algebra)

- Why is this a reasonable definition? I have no idea. We'll have to take this as a black box. ■ (The interested reader should consult [AG, Diff'l Operators on loop groups via Chiral algebras, Appendix])

- But it works:

  - For $K \subset G((t))$ s.t. $G((t))/K$ is an ind-scheme,

$$D_{G((t))/K}\text{-modules} \simeq \left(D_{G((t))}\text{-modules}\right)^K$$

via our previous definitions — [AG], [BD] — chiral algebra definition

In all Frenkel-Gaitsgory papers, this is the definition of D-modules they use.

Okay, now we finally have some idea of what D-modules on $\mathcal{Fl}$, Gr are. So what is known about them?

note: on $\mathcal{Fl}$ and Gr, can twist $D(\mathcal{Fl})$, $D(Gr)$ by level, obtaining "TDO"s $D(\mathcal{Fl})_\kappa$, $D(Gr)_\kappa$

The state of affairs, as far as I can gather:

exactness results:

$\Gamma: D(Gr)_\kappa \longrightarrow \mathfrak{g}_\kappa\text{-mod}$ is
- exact when $\kappa + 1/2 \in \mathbb{Q}^{<0}$ (negative) or $\kappa \notin \mathbb{Q}$ (irrational) [BD Thm 7.15.8] or [FG 04 (Duke)]
  methods similar to f.d. setting
- exact when $\kappa = -1/2$ (critical) [FG04, Duke]
  chiral algebra methods
- not exact when $\kappa + \frac{1}{2} \in \mathbb{Q}^{>0}$ (positive) [BD?]

$\Gamma: D(\mathcal{Fl})_\kappa \longrightarrow \mathfrak{g}_\kappa\text{-mod}$ is
- exact at negative or irrational level [BD Thm 7.15.8]
- not exact at critical level [FG 09, Rep'n Theory]
  but can move to derived categories and it's fine
- not exact at positive level [BD?]

can replace $\mathcal{Fl}$ with $\widetilde{\mathcal{Fl}} = G((t))/I_u$, $I_u = ev_0^{-1}(N)$ and all results still hold

fully-faithful results:

$\Gamma: D(Gr)_\kappa \longrightarrow \mathfrak{g}_\kappa\text{-mod}$ is
- fully faithful at negative or irrational [BD, §7]
- faithful at critical level [FG 09 Annals] but not full
- ?? positive level

$\Gamma: D(\mathcal{Fl})_\kappa \longrightarrow \mathfrak{g}_\kappa\text{-mod}$ is
- fully faithful at negative or irrational [BD, §7]
- not fully faithful at critical level, even at derived level [FG09, RepTheory] (9)
  can "fix" this by changing domain category to $D^b(Coh(\mathcal{O}_p{}^{nilp})) \otimes D^b(D(\mathcal{Fl})\text{-mod})$
  $D^b(Coh(\widetilde{\mathcal{N}}/\check{G}))$ ← [Arkhipov-Bezrukavnikov]
- ?? positive level

essential surjectivity results:

- Γ is never essentially surjective. But there is progress in this direction:
  - κ negative: $D(\widetilde{\mathcal{Fl}})_\kappa\text{-mod}^{I_u, T, w, \lambda} \xrightarrow{\sim} (\mathcal{O}_{aff})_{\nu(\lambda)}$ proven in [FG09, Annals, appendix] attributed to [KT Neg level I]
    I_u-equt, weakly T-equt, λ-twisted T-action; block of category $\mathcal{O}$, W-orbit of λ
    so negative level category $\mathcal{O}$ is in the essential image
- for integral κ, Beilinson has a conjectural description of essential image of $\Gamma: D(\widetilde{\mathcal{Fl}})_\kappa\text{-mod} \longrightarrow \mathfrak{g}_\kappa\text{-mod}$ but I don't understand it so I can't explain it.
  (see [Beilinson 06, Langlands parameters for Heisenberg modules] arXiv version, NOT published version) intro Rmk (ii)
- critical level:
  [FG04 Duke] $\Gamma: D(Gr)_{crit}\text{-mod} \longrightarrow \mathfrak{g}_{crit}\text{-mod}_{reg}$ = $\mathfrak{g}_{crit}$-modules on which $\mathfrak{z}_{\mathfrak{g}}$ = center of completed enveloping algebra of $\hat{\mathfrak{g}}_{crit}$ acts as it does on the vacuum module $\mathbb{V}_{crit} := \mathrm{Ind}_{\mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}_{crit}}(\mathbb{C})$
  problem: not full
  fix: take "Hecke eigenobjects" in $D(Gr)_{crit}$
  [FG09 annals] $\Gamma: D(Gr)^{Hecke}_{crit\text{-mod}} \longrightarrow \mathfrak{g}_{crit}\text{-mod}_{reg}$ conjecture [FG]: This is an equivalence
  NOT PROVEN

NOT PROVEN

Special case: add I_u-equivariance:

[FG09 Annals] $\Gamma : D(Gr)^{\text{Hecke}}_{\text{crit}}\text{-mod}^{I_u} \xrightarrow{\sim} \mathfrak{g}_{\text{crit}}\text{-mod}^{I_u}_{\text{reg}}$ equivalence

Finally: A sketch of the negative level equivalence

$$\underset{\substack{\uparrow \\ I_u\text{-equt}}}{D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}}^{\overset{\text{weakly T-equt}}{I_u, T, w}, \overset{\lambda\text{-twisted T-action}}{\lambda}} \xrightarrow{\sim} (\mathcal{O}_{aff})_{\nu(\lambda)}$$

block of category $\mathcal{O}$; W-orbit of λ

$p^* \mathcal{F} \xrightarrow{\sim} act^* \mathcal{F}$ iso of $\underline{\mathcal{O}}$-modules (strong = iso of D-modules)

First:

- What are these categories?
  - $\widetilde{\mathcal{F}\ell} = G((t))/I_u$ $\circlearrowleft T = I/I_u$ right action $\rightsquigarrow$ $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T,w}$ = weakly T-equt objects in $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}$
  - $\Gamma^T : D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod} \longrightarrow \hat{\mathfrak{g}}_\kappa\text{-mod}$ composition of Γ and T-invariants
  - objects in $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T,w}$ have a $S(\mathfrak{h})$-action $a^\#: S(\mathfrak{h}) \longrightarrow End(\mathcal{F})$
  - define for $\lambda \in \mathfrak{h}^*$ $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T,w,\lambda}$ = objects in $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T,w}$ where $(a^\#(h) - \lambda(h))$ acts nilpotently $\forall h \in \mathfrak{h}$
  - Finally, $D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T,w,\lambda,I_u}$ = I_u-equt objects in $\mathcal{F}$ (I_u and T connected $\Rightarrow$ this is a full subcategory of $D(\widetilde{\mathcal{F}\ell})$-mod)
  - Let $\mathcal{O}_{aff} := \hat{\mathfrak{g}}_\kappa\text{-mod}^{I_u}$ (note: slightly different category $\mathcal{O}$ than Arun / Yapings)

    $\cup$ $\quad \nu$ = W-orbit in $\mathfrak{h}^*$

    $\mathcal{O}_\nu$ = full subcat of objects admitting a filt'n w/ subquotients $L_{\kappa,\mu}$ for $\mu \in \nu \in W \backslash \mathfrak{h}^*$

<u>Theorem</u> ([FG09, Annals, Thm 5.5], attribute argument to [BD] + [KT]) For $\kappa + \frac{1}{2} \notin \mathbb{Q}^{\geq 0}$ (negative level), λ s.t. $M(\lambda)$ is irreducible (antidominant)

$$\Gamma^T : D(\widetilde{\mathcal{F}\ell})_\kappa\text{-mod}^{T, w, \lambda, I_u} \xrightarrow{\sim} (\mathcal{O}_{aff})_{\nu(\lambda)}$$

is an equivalence.

<u>Sketch of proof</u>

- By analogous argument to f.d. argument sketched earlier, Γ^T is exact
- A computation in [KT, negative level I] gives:

$$\Gamma(\widetilde{\mathcal{F}\ell}, j_{w,\lambda,*}) \simeq M^\vee(\kappa, w \cdot 0) \qquad \Gamma(\widetilde{\mathcal{F}\ell}, j_{w,\lambda,!}) \simeq M(\kappa, w \cdot 0)$$

($\widetilde{\mathcal{F}\ell} \longrightarrow \mathcal{F}\ell = \bigsqcup_{w \in W} X_w$. $\exists!$ I_u-equt irred twisted $D(\widetilde{\mathcal{F}\ell})$-module on pre-image of X_w in $\widetilde{\mathcal{F}\ell}$. These are the $*$ and ! extensions of that.)

- To prove fully faithful, reduce to std D-module supported on a point, use KT computation + known facts about Homs btwn dual Vermas + Vermas
- To show essential surjectivity, use

① <u>lemma:</u> $G : \mathcal{C}_1 \to \mathcal{C}_2$ exact functor on abelian satisfying (i) G fully faithful (ii) $\mathrm{Ext}^1_{\mathcal{C}_1}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{C}_2}(GX, GY)$
[lem 3.10 FG, Annals] $\Rightarrow$ if G admits <u>conservative right adjoint</u>, G is an equivalence

② <u>lemma</u>: Γ^T admits a right adjoint $F: (\mathcal{O}_{\text{aff}})_{\nu(\lambda)} \longrightarrow D(\widetilde{\mathcal{F}\ell})_\kappa^{T,\omega,\lambda,I_n}$-mod
[FG Annals, Lemma 3.12]

(can actually show $\Gamma: D(Gr)_\kappa$-mod $\longrightarrow \mathfrak{g}_\kappa$-mod admits a right adjoint for any level κ)

F is conservative (i.e. $M \neq 0 \Rightarrow FM \neq 0$) b/c for any nonzero $M \in \mathcal{O}$, $\exists$ nonzero map

$$\overset{\text{some Verma}}{M(\mu)} \longrightarrow M$$

$$\Rightarrow \overset{\neq 0}{\mathrm{Hom}_{\mathcal{O}}}(M(\mu), M) = \mathrm{Hom}(\Gamma(\widetilde{\mathcal{F}\ell}, j_{w!\lambda}), M) = \mathrm{Hom}(j_{w!\lambda}, FM)$$

$\square$

That's all!