Kazhdan-Lusztig Conjecture for Kac-Moody Lie algebras

Ref: Kashiwara-Tanisaki
Kac's book

I) Kac-Moody Lie algebras
II) Cat. $\mathcal{O}$
III) KL Conjecture.

Generalized Cartan Matrix

$A = (a_{ij})$ $\quad i, j = 0, 1, \cdots n$ $\quad$ s.t
$a_{ij} \in \mathbb{Z}$

① $a_{ii} = 2$
② $a_{ij} \leq 0$, $i \neq j$
③ $a_{ij} = 0 \Rightarrow a_{ji} = 0$

A realization of A is $(\mathfrak{h}, \Pi, \Pi^\vee)$ (There exists a unique $/\cong$).

- $\mathfrak{h}$ $\mathbb{C}$-vector space $\dim \mathfrak{h} = 2(n+1) - \mathrm{rk} A$
- $\Pi = \{\alpha_0, \alpha_1, \cdots \alpha_n\} \subseteq \mathfrak{h}^*$ linearly indep simple roots
- $\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \cdots \alpha_n^\vee\} \subseteq \mathfrak{h}$ linearly indep simple coroots

s.t. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Affine type: A indecomposible
corank(A) = 1.

e.g. $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

finite dim type: $\det(A) \neq 0$.

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- A is symmetrizable, if $\exists\ D = \begin{bmatrix} \varepsilon_0 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix}$

s.t. $A = D\,B$ & symmetric

$\Rightarrow \exists$ a symmetric bilinear $\mathbb{C}$-valued form on $\mathfrak{h} = \sum_{i=0}^{n} \mathbb{C}\alpha_i^\vee \oplus \mathfrak{h}''$

$(\cdot|\cdot) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}_{\in \mathfrak{h}^* \in \mathfrak{h}}$

$(\alpha_i^\vee | h) := \langle \alpha_i, h \rangle \varepsilon_i \ , \ i = 0, 1, \cdots n$ — Still non-deg on $\mathfrak{h}$

$(h' | h'') = 0, \quad h', h'' \in \mathfrak{h}''$ — (Lemma 2.1 of Kac)

$\tilde{\mathfrak{g}}(A)$ with generators $e_0, e_1, \cdots e_n$ satisfies the KM presentation.

$f_0, f_1, \cdots f_n$

& $\mathfrak{h}$

i.e.: ① $[e_i\ f_j] = \delta_{ij}\ \alpha_i^\vee \quad (i,j = 0, 1, \cdots n)$

$[h, h'] = 0 \quad (h, h' \in \mathfrak{h})$

$[h\ e_i] = \langle \alpha_i, h \rangle e_i$

$[h\ f_j] = - \langle \alpha_i\ h \rangle f_i \quad (i = 0, 1, \cdots n,\ h \in \mathfrak{h})$

$\left.\right\} \tilde{\mathfrak{g}}(A)$

② $\mathfrak{h} \subseteq \tilde{\mathfrak{g}}(A)$ embedding

$r :=$ the max ideal in $\tilde{\mathfrak{g}}(A)$ that intersects $\mathfrak{h}$ trivially.

$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A) / r.$

A indecomposible:

$\det(A) \neq 0 \longrightarrow \mathfrak{g}(A)$ f. dim simple

$\operatorname{corank}(A) = 1 \longrightarrow \mathfrak{g}(A)$ affine

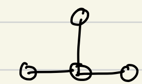
$\operatorname{corank}(A) > 1 \longrightarrow \mathfrak{g}(A)$ indefinite type.

non-twisted $\qquad$ twisted.

For non-twisted affine Lie alg. ([Kac, chapter4] $\boxed{X_n^{(1)}}$ $\quad \underbrace{X_n^{(2)} \quad X_n^{(3)}}$)

twised $\rightsquigarrow$ σ: non-trivial diag. automorphism of $\mathring{\mathfrak{g}}$.

If $|\sigma| = 2$. $\quad \mathfrak{g}$ = type A_n, D_n, E_6

If $|\sigma| = 3$. $\quad \mathfrak{g}$ = type D_4. $\quad D_4$: 

Loop realization:

$$\mathfrak{g}(A) = \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$ bracket:

- c is central
- $[d, Xt^k] = k\, X t^k$. $\quad$ (d acts as : $t \frac{d}{dt}$, and it kills c)
- $[Xt^k, Yt^l] = [X\, Y]\, t^{k+l} + k\, \delta_{k,-l} \langle X, Y \rangle c$.

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$\mathfrak{h}$ dim $n+2$ basis $\{\alpha_0^\vee, \alpha_1^\vee, \cdots \alpha_n^\vee, d\}$, fix d s.t $\langle \alpha_i, d\rangle = 0$, $i=1,\cdots n$

$\langle \alpha_0, d\rangle = 1$.

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus (\mathbb{C}c + \mathbb{C}d)$$

$\overset{\circ}{\mathfrak{h}}$: dim n; c: central elt

$$\alpha_0^\vee = \frac{2}{(\theta,\theta)} c - \theta^\vee$$
$$= c - \theta^\vee$$

Define $\delta \in \mathfrak{h}^*$ s.t: $\delta|_{\overset{\circ}{\mathfrak{h}} + \mathbb{C}c} = 0$, $\langle \delta, d\rangle = 1$

ie: $\langle \delta, \alpha_i^\vee\rangle = 0$ $i=0,\cdots n$

$\langle \delta, c\rangle = 0$

$\langle \delta, d\rangle = 1$

$\langle \Lambda_0, \alpha_0^\vee\rangle = 1$

$\langle \Lambda_0\ \alpha_i^\vee\rangle = 0$ $i=1,2,\cdots n$

$\langle \Lambda_0, d\rangle = 0$.

Then:

$$\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}{}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)$$
$$= \mathbb{C}\langle \alpha_0, \cdots \alpha_n\rangle + \mathbb{C}\Lambda_0.$$

$$\delta = \alpha_0 + \theta.$$

θ: h. root of $\overset{\circ}{\mathfrak{g}}$

$$e_i \longleftrightarrow E_i \otimes t^0 \qquad i=1,\cdots n. \qquad e_0 \longleftrightarrow \underset{\in}{F_\theta} \otimes t \quad \mathring{\mathfrak{g}}_{-\theta}$$

$$f_i \longleftrightarrow F_i \otimes t^0 \qquad f_0 \longleftrightarrow \underset{\in \mathring{\mathfrak{g}}_\theta}{E_\theta} \otimes t^{-1}$$

Triangular decomposition. n_- gen. by $f_0, f_1, \cdots$ $\quad n_+$ gen. by $e_0, e_1, \cdots$

$$\mathfrak{g}(A) = \left(\bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha\right)$$

$$\underset{\text{root}}{\mathfrak{g}_\alpha} = \{ x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x, \ \forall h \in \mathfrak{h} \}$$

$$\mathfrak{g}_0 = \mathfrak{h}$$

root system:

$$\Delta = \underbrace{\{ \alpha + i\delta \mid \alpha \in \mathring{\Delta}, \ i \in \mathbb{Z} \}}_{= \Delta^{real}} \cup \underbrace{\{ j\delta \mid j \in \mathbb{Z}\setminus\{0\}\}}_{= \Delta^{im}}$$

&

$$\mathfrak{g}_{\alpha+i\delta} = \mathring{\mathfrak{g}}_\alpha \otimes t^i$$

$$\mathfrak{g}_{j\delta} = \mathring{\mathfrak{h}} \otimes t^j \qquad j \neq 0.$$

$$\Delta_+^{real} = \{\alpha + i\delta \mid \alpha \in \mathring{\Delta}, i > 0\} \cup \mathring{\Delta}_+.$$

$$\Delta_+^{Im} = \{n\delta \mid n \in \mathbb{N}\} = \{\delta, 2\delta, 3\delta, \cdots\}_{n \in \mathbb{N}}.$$

$$\mathfrak{n}_+ = \mathbb{C}[t] \otimes \mathring{\mathfrak{n}}_+ + \left(t\,\mathbb{C}[t] \otimes (\mathring{\mathfrak{n}}_- + \mathring{\mathfrak{h}})\right)$$

$$\mathfrak{n}_- = \mathbb{C}[t^{-1}] \otimes \mathring{\mathfrak{n}}_- + \left(t^{-1}\mathbb{C}[t^{-1}] \otimes (\mathring{\mathfrak{n}}_+ + \mathring{\mathfrak{h}})\right)$$

Affine Weyl gp:

fund. reflection. $\mathfrak{h}^* \longrightarrow \mathfrak{h}^*$

r_i: $\lambda \longmapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$

$$W \underset{\text{sub gp}}{\subseteq} GL(\mathfrak{h}^*)$$

$$\| $$

$$\{r_0, r_1, \cdots r_n\}$$

Recall $\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)$.

& $\langle \delta, \alpha_i^\vee \rangle = 0 \quad i = 0, 1, \cdots \ell. \implies w(\delta) = \delta. \ \forall w \in W.$

$\mathring{W} = \langle r_1, r_2, \ldots r_n \rangle$ finite Weyl gp.

$\langle \Lambda_0, \alpha_i^\vee \rangle = 0, \ i = 1, \cdots \ell \implies \mathring{W}$ acts trivially on $\mathbb{C}\Lambda_0 + \mathbb{C}\delta$.

Prop: $W = \mathring{W} \ltimes \mathring{Q}^\vee$ coroot lattice $\qquad \mathring{Q}^\vee \leq \mathfrak{h}$

$$\mathring{Q}^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i^\vee.$$

$\forall \alpha$ real root, $\quad r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \ \lambda \in \mathfrak{h}^*.$

highest root $\theta = \delta - \alpha_0$

$r_\theta, r_{\alpha_0} \in W, \quad \boxed{r_{\alpha_0} r_\theta = t_{\theta^\vee}}$ $\in \mathring{Q}^\vee$

For $\alpha \in \mathring{Q}^{\vee}$, $\underbrace{\text{if } \langle \lambda, c \rangle = 0}_{\Updownarrow}$, then: $\mathfrak{h}^* \longrightarrow \mathfrak{h}^*$
$t_\alpha(\lambda) = \lambda - (\lambda|\alpha)\delta.$

Recall: $\left(\lambda \in \bigoplus_{i=0}^{n} \mathbb{C}\alpha_i\right)$

$\langle c, \alpha_i \rangle = 0 \quad i = 0, 1, \cdots n$

$\langle c, \delta \rangle = 0$

$\langle c, \Lambda_0 \rangle = 1$

A finite type	A affine KM type
$\iff \mathfrak{g}(A)$ simple f. dim Lie alg	• $\mathfrak{g}(A)$ inf. dim.
$\iff (\cdot\mid\cdot)_{\mathfrak{h}_{\mathbb{R}}}$ is positive definite	• $(\cdot\mid\cdot)_{\mathfrak{h}_{\mathbb{R}}}$ is no longer positive def.
$\iff \lvert W \rvert < \infty$	• $\lvert W \rvert = \infty$.
$\iff \lvert \Delta \rvert < \infty$	• $\lvert \Delta \rvert = \infty$.
$\iff \Delta^{im} = \phi$	• $\exists$ imaginary roots

Example of $\widehat{sl}_2$:

$$A_1^{(1)} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

two simple roots α_0, α_1

$\delta = \alpha_0 + \alpha_1$

$$\Delta_+ = \Delta_+^{real} \sqcup \Delta_+^{im}$$

$$\Delta_+^{real} = \{\alpha_1, \pm\alpha_1 + n\delta \mid n > 0\}$$

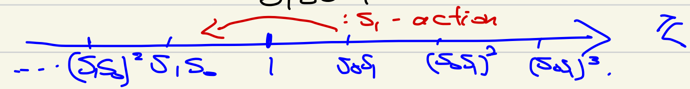
$$\Delta_+^{im} = \{n\delta \mid n = 1, 2, 3, \cdots\}$$

$\delta = \alpha_0 + \alpha_1$

$$W = \langle s_0, s_1 \mid s_0^2 = 1, s_1 = 1 \rangle \ni s_0 s_1 s_0 s_1 \cdots, \quad s_1 s_0 s_1 s_0 \cdots$$

$$= \langle s_1 \rangle \ltimes \mathbb{Z}\langle s_0 s_1 \rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}$$

with $s_1 (s_0 s_1) s_1^{-1} = s_1 s_0$.



$$\mathfrak{h}^* = \mathbb{C}\delta + \mathbb{C}\alpha_1 + \mathbb{C}\Lambda_0$$

$\delta = \alpha_0 + \alpha_1$

since $\langle \Lambda_0, \alpha_1^\vee \rangle = 0$

$W \curvearrowright \mathfrak{h}^*$,

$s_1(\delta) = \delta$, $s_1(\alpha_1) = -\alpha_1$, $s_1(\Lambda_0) = \Lambda_0$

$s_0(\delta) = \delta$, $s_0(\alpha_1) = 2\delta - \alpha_1$, $s_0(\Lambda_0) = -\delta + \alpha_1 + \Lambda_0$

$s_1(\omega_1) = -\omega_1$

$s_0(\omega_1) = \delta - \omega_1$

Since: $s_0(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_0^\vee \rangle \alpha_0$

$= \alpha_1 + 2\alpha_0$

$= \alpha_1 + 2(\delta - \alpha_1)$

$= 2\delta - \alpha_1$

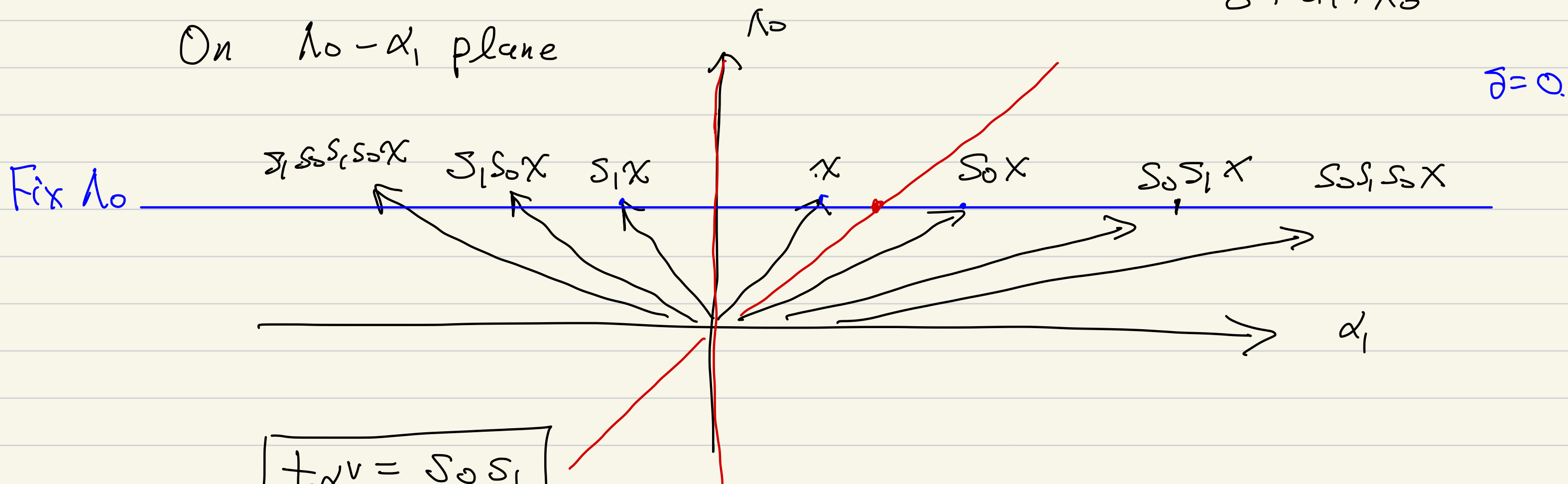
$s_0(\Lambda_0) = \Lambda_0 - \langle \Lambda_0, \alpha_0^\vee \rangle \alpha_0$

$= \Lambda_0 - \alpha_0$

$= \Lambda_0 - (\delta - \alpha_1)$

$= -\delta + \alpha_1 + \Lambda_0$

On $\Lambda_0 - \alpha_1$ plane



$\boxed{t_{\alpha^\vee} = s_0 s_1}$

$s_1(\Lambda_1) = s_1(\Lambda_0 + \frac{\alpha_1}{2})$

$= \Lambda_0 + (-\frac{\alpha_1}{2})$

$= \Lambda_0 - (\Lambda_1 - \Lambda_0)$

$= 2\Lambda_0 - \Lambda_1$

$s_0(\Lambda_1) = s_0(\Lambda_0 + \frac{\alpha_1}{2})$

$= -\delta + \alpha_1 + \Lambda_0 + \frac{1}{2}(2\delta - \alpha_1)$

$= \frac{\alpha_1}{2} + \Lambda_0$

$= \Lambda_1$

$s_0(\Lambda_0) = -\delta + \alpha_1 + \Lambda_0$

$= -\delta + 2(\Lambda_1 - \Lambda_0) + \Lambda_0$

$= 2\Lambda_1 - \Lambda_0 - \delta$

In terms of the fundamental weights

$$\mathfrak{h}^* = \mathbb{C}\delta + \mathbb{C}\Lambda_0 + \mathbb{C}\Lambda_1,$$

where: $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. $\Rightarrow$ $\Lambda_1 = \Lambda_0 + \omega_1 = \Lambda_0 + \frac{\alpha_1}{2}$.

Set $\delta = 0$ $\Rightarrow$

$s_1(\Lambda_1) = 2\Lambda_0 - \Lambda_1$ $\qquad$ $s_0(\Lambda_1) = \Lambda_1$

$s_1(\Lambda_0) = \Lambda_0$ $\qquad$ $s_0(\Lambda_0) = -\Lambda_0 + 2\Lambda_1 - \delta$

$s_0 s_1(\Lambda_1) = -2\Lambda_0 + 3\Lambda_1$ $\qquad$ $s_1 s_0(\Lambda_1) = 2\Lambda_0 - \Lambda_1$

$s_0 s_1(\Lambda_0) = -\Lambda_0 + 2\Lambda_1$ $\qquad$ $s_1 s_0(\Lambda_0) = 3\Lambda_0 - 2\Lambda_1$

$s_1 s_0 s_1(\Lambda_1) = 4\Lambda_0 - 3\Lambda_1$ $\qquad$ $s_0 s_1 s_0(\Lambda_1) = -2\Lambda_0 + 3\Lambda_1$

$s_1 s_0 s_1(\Lambda_0) = 3\Lambda_0 - 2\Lambda_1$ $\qquad$ $s_0 s_1 s_0(\Lambda_0) = -3\Lambda_0 + 4\Lambda_1$

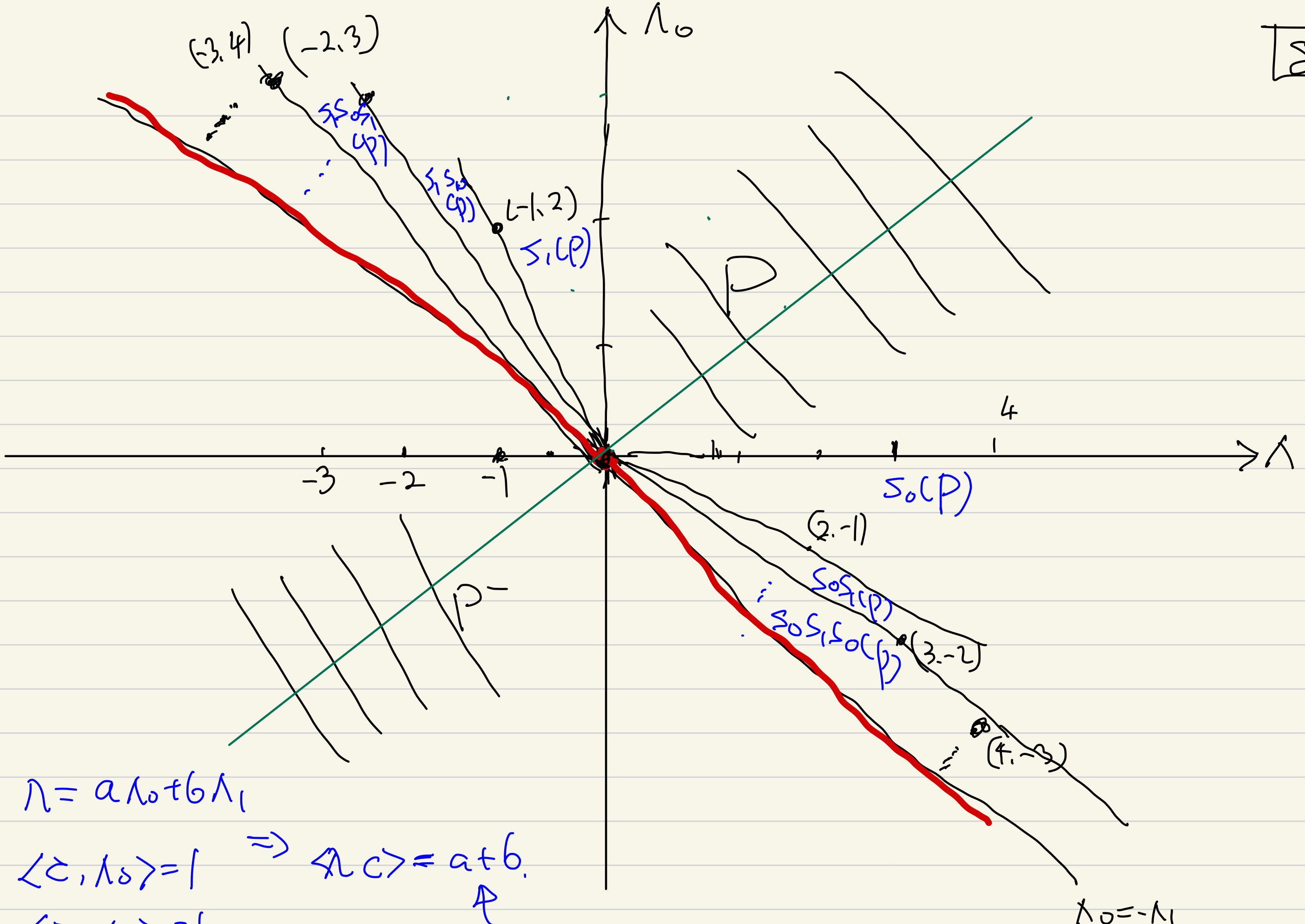
$\vdots$

$s_1(\Lambda_0 - \Lambda_1) = -\Lambda_0 + \Lambda_1$ $\qquad$ $s_0(\Lambda_0 - \Lambda_1) = -\Lambda_0 + \Lambda_1$.

$\Rightarrow$ W fixes $\underline{\Lambda_0 - \Lambda_1}$.

Wall:

$\boxed{\delta = 0}$



$\Lambda = a\Lambda_0 + b\Lambda_1$

$\langle c, \Lambda_0 \rangle = 1$
$\langle c, \Lambda_1 \rangle = 1$
$\Rightarrow \langle \Lambda, c \rangle = a + b$

↑ level.

- **About level & Center of $U(\mathfrak{g})$:**

c central elt

$\tilde{U}(\mathfrak{g})_k := \tilde{U}(\mathfrak{g}) / (k\cdot 1 - c)$ level k quotient

1 unit of $\tilde{U}(\mathfrak{g})$

the completion of $U(\mathfrak{g})$ w.r.t the topology given by $U(\mathfrak{g}[[t]])t^n$

Feigin Frenkel: $\mathfrak{Z}_{\mathfrak{g}} := \text{Center}(\tilde{U}_k(\hat{\mathfrak{g}}))$

Recall: $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 0, 1, \cdots n$. $\rho \in \mathfrak{h}^*$

If $k \neq -h^\vee$, $\mathfrak{Z}_{\mathfrak{g}} = \mathbb{C}\cdot 1$,

If $k = -h^\vee$ critical level, $\text{Spec}(\mathfrak{Z}_{\mathfrak{g}}) = \text{Op}_{\check{G}}(D^\times)$ the ind-scheme of $\check{G}$-opers on the formal punctured disc

$h^\vee$: dual Coxeter number for $\mathring{\mathfrak{g}}$.

$\| \langle c, \rho \rangle$

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What's $h^\vee$?

$$\theta^\vee = \sum_{i=1}^{n} c_i \alpha_i^\vee$$

$$h^\vee := 1 + \sum_{i=1}^{n} c_i .$$

Recall $\alpha_0^\vee = \frac{2}{(\theta,\theta)} c - \theta^\vee$.

$$c = \alpha_0^\vee + \theta^\vee$$

$$\langle c, \rho \rangle = \langle \alpha_0^\vee, \rho \rangle + \langle \sum_{i=1}^{n} c_i \alpha_i^\vee, \rho \rangle$$

$$= 1 + \sum_{i=1}^{n} c_i$$

$$= h^\vee .$$

Def: The cat $\mathcal{O}$ is the cat. of $\mathfrak{g}$-mod V s.t

1. V is $\mathfrak{h}$-diagonalizable
2. the weight spaces of V are f. dim
3. $\exists$ finitely many elts $\mu_1, \cdots \mu_k \in \mathfrak{h}^*$, s.t $\forall$ weight μ of V,
   $\Rightarrow \mu \leq \mu_{i_0}$ for some i.

That is: $\mu_i - \mu = \sum_{0 \leq i \leq n} m_i \alpha_i$, for some $m_i \in \mathbb{Z}_{\geq 0}$.

A representation V is of <u>level k</u>, if c acts as $k\,\mathrm{Id}$ on V.

Lemma (Schur Lemma)

L simple reps. $\in \mathcal{O}$ $\Rightarrow$ a central elt acts as a scalar

<u>Pf</u>: c eigenvalue k.

$\ker(c - k\,\mathrm{Id}) \underset{\text{submod}}{\subseteq} L$

$\Rightarrow L = \ker(c - k\,\mathrm{Id})$.

In [KT]: $\mathcal{O}'$ = the cat of left $\mathfrak{g}$-mods M s.t

- M is a generalized wt mod as $\mathfrak{h}$-mod

  ie: $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, $M_\mu = \{ u \in M \mid \exists\ n \in \mathbb{Z}_{\geq 0}$, s.t $(h - \mu(h))^n u = 0 \ \forall h \in \mathfrak{h} \}$

  generalized eigen space

  $\dim(M_\mu) < \infty$

- $\exists\ \mu_1, \dots \mu_k \in \mathfrak{h}^*$, s.t

  $M_\mu \neq 0 \Rightarrow \mu \leq \mu_i$ for some i.

In [Frenkel-Ben Zvi]

trig decomp: $\left(\mathring{\mathfrak{g}} \otimes t^{-1}\mathbb{C}[t^{-1}]\right) \oplus \mathring{\mathfrak{g}} \oplus \left(\mathring{\mathfrak{g}} \otimes t\,\mathbb{C}[t]\right) \oplus \mathbb{C}c$,

t^s

$s > 0$

$s \to s < 0$

$\mathcal{O}_k :=$ the cat. of mods of the cat $\mathcal{O}$ of level k.

$\nwarrow$ not stable under $\otimes$. since $V_1 \in \mathcal{O}_{k_1}$, $V_2 \in \mathcal{O}_{k_2}$ $\quad V_1 \otimes V_2 \in \mathcal{O}_{k_1+k_2}$

as $\Delta(c) = c \otimes 1 + 1 \otimes c$.

- $\forall \lambda \in \mathfrak{h}^*$.

Verma mod with h wt λ.

$$M(\lambda) = U(\mathfrak{g}) \Big/ \Big(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})\,(h - \lambda(h)1) + U(\mathfrak{g})\mathfrak{n}^+ \Big)$$

Prop:

$M(\lambda) \in \mathcal{O}$.

$\cong$

$U(\mathfrak{n}^-)$

Pf: $\circ$ $\forall \mu$ wt of $M(\lambda)$, $\mu \leq \lambda$. $\Rightarrow$ ③

$\circ$ $\mu = \lambda - \sum_{i=0}^{n} m_i \alpha_i$. $\quad m := \sum_{i=0}^{n} m_i$

$$M(\lambda)_\mu \subseteq \sum_{(i_1, \cdots, i_m) \in \{0,1,\cdots,n\}^m} \mathbb{C} F_{i_1} F_{i_2} \cdots F_{i_m} \Rightarrow \text{f. dim} \Rightarrow ②$$

c acts as $\overset{\in\mathfrak{h}^*}{\lambda}(\overset{\in\mathfrak{h}}{c})$ on $M(\lambda)$.

Character: $ch(M) := \sum_{\mu\in\mathfrak{h}^*} \dim M_\mu \, e^\mu$.

$$= e^\lambda \Big/ \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}.$$

$U(\mathfrak{n}_-)$ basis $\{\overset{\in\mathfrak{g}_{-\alpha}}{X_{-\alpha}} \mid \alpha\in\Delta_+\}$

Let: $L(\lambda)$ = irreducible quotient of $M(\lambda)$. level $k=\lambda(c)$.

Problem: Determine $ch\,L(\lambda)$.

Part III KL Conj

Fix $\rho \in \mathfrak{h}^*$ s.t. $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 0, 1, \cdots n$

Use the shifted action: $W \curvearrowright \mathfrak{h}^*$

$$w \circ \mu := w(\mu + \rho) - \rho.$$

$$\Rightarrow \quad w \circ (-\rho) = w(0) - \rho = -\rho.$$

w fixes $-\rho$.

Weight lattice $P := \{ \langle x, \alpha_i^\vee \rangle \in \mathbb{Z}.\ \forall i = 0, 1, \cdots n \}$

$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ e.g. $\begin{cases} \Lambda_0 \\ \Lambda_1 = \Lambda_0 + \omega_1 \end{cases}$

Fund. weights.

$$P^+ = \{ \lambda \in P \mid \langle \lambda + \rho, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0},\ i = 0, 1, \cdots n \}$$

$$P^- = \{ \lambda \in P \mid \langle \lambda + \rho, \alpha_i^\vee \rangle \in \mathbb{Z}_{<0}\ \ i = 0, 1, \cdots n \}$$

Rmk. Use $c = \alpha_0^\vee + \theta^\vee$. $\langle \lambda + \rho, c \rangle > 0$ positive level.

$\lambda \in P^+$

P^- $\langle \lambda + \rho, c \rangle < 0$ negative level.

$\langle \lambda+\rho, c\rangle = 0$ critical level.

What's the level of $w \circ \lambda$? $\quad$ w fixes c, since $\langle c, \alpha_i\rangle = 0$. $i = 0, \dots n$.

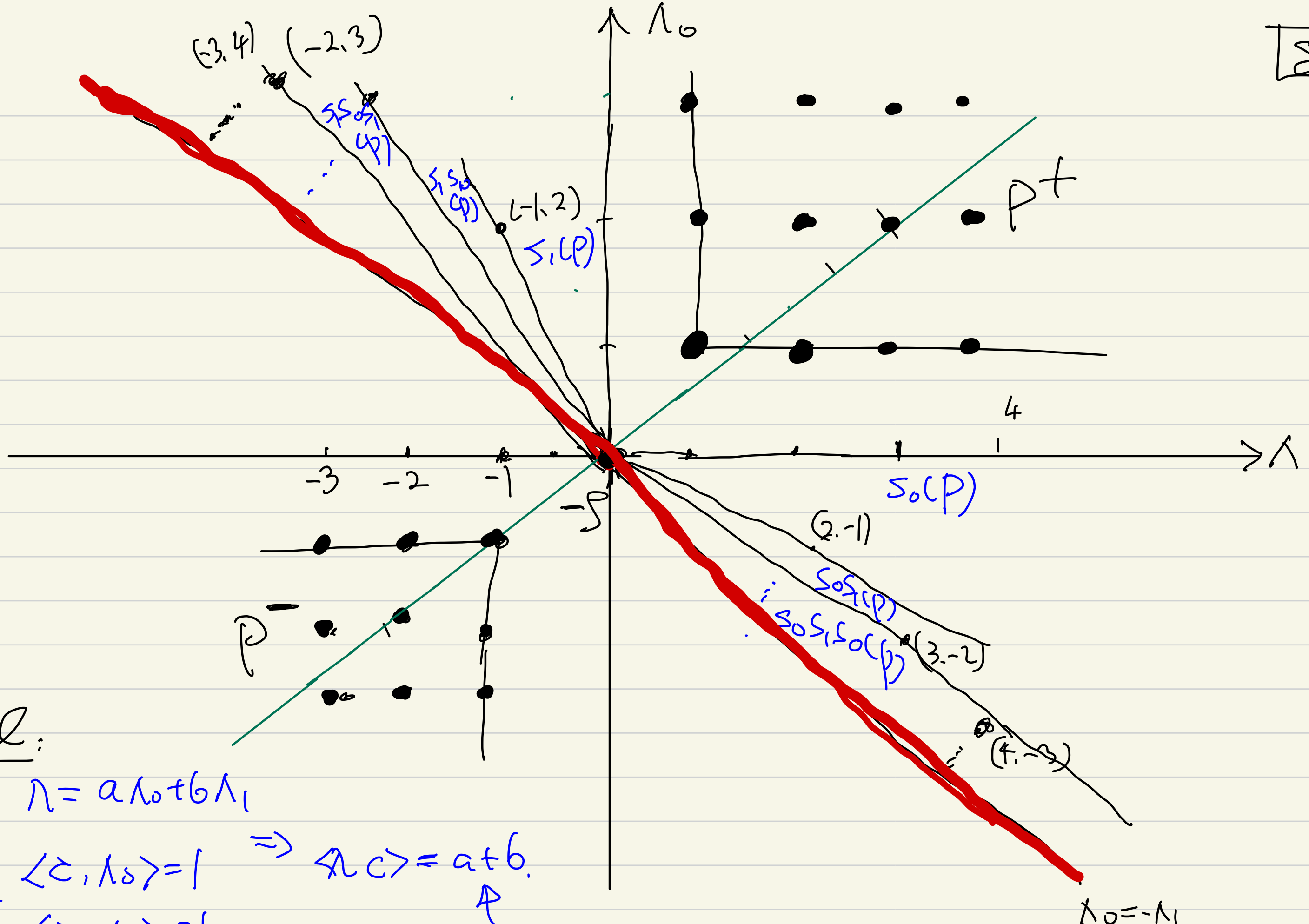
$$\langle w\circ\lambda, c\rangle = \langle w(\lambda+\rho)-\rho, c\rangle = \langle w(\lambda+\rho), c\rangle - \langle\rho, c\rangle$$

$$\langle \lambda'. c\rangle \qquad\qquad = \langle \lambda+\rho, c\rangle - \langle \rho. c\rangle = \langle \lambda\ c\rangle$$

level under $W\circ$ action won't change. $\langle w\circ\lambda\ c\rangle = \langle\lambda. c\rangle$.

Wall:

$\boxed{\delta=0}$ 22



Level:

$\Lambda = a\Lambda_0 + b\Lambda_1$

$\langle c, \Lambda_0\rangle = 1$

$\langle c, \Lambda_1\rangle = 1$

$\Rightarrow \langle \Lambda, c\rangle = a+b.$

level.

Recall: KS polynomial for Coxeter system (W, S), $S = \{r_0, r_1, \dots r_n\}$

Hecke alg: $H(W)$ = free $\mathbb{Z}[q, q^{-1}]$-mod with basis $\{T_w \mid w \in W\}$

Algebra str.: $T_{w_1} T_{w_2} = T_{w_1 w_2}$ if $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$

$(T_s + 1)(T_s - q) = 0 \quad s \in S.$

What's length? $w = r_{i_1} \cdots r_{i_s} \in W$ reduced expression $\ell(w) = s.$

Prop (KL) $\forall w \in W$, $\exists$ an elt $C_w \in H(W)$ of the form

$$C_w = \sum_{y \leq w} P_{yw}(q) T_y \quad (P_{yw}(q) \in \mathbb{Z}[q])$$

s.t.: ① $P_{ww} = 1$

② $\deg P_{yw} \leq 1/2 \, (\ell(w) - \ell(y) - 1)$, $y \leq w$.

③ $\overline{C_w} = q^{-\ell(w)} C_w$ ← bar involution

ie: $\overline{q} = q^{-1}$

& $\overline{T_w} = T_{w^{-1}}^{-1}$, $w \in W$.

Ex: $\widehat{sl_2}$:

Let $w = s_1 s_2 \cdots s_q$ reduced word for w.

Positive level Bruhat order

$x \le w \iff$ · x has a reduced word
· subword of w.

$$
\begin{tikzcd}
 & 1 & \\
s_0 \arrow[ur] & & s_1 \arrow[ul] \\
s_0 s_1 \arrow[u] \arrow[urr] & & s_1 s_0 \arrow[u] \arrow[ull] \\
s_0 s_1 s_0 \arrow[u] \arrow[urr] & & s_1 s_0 s_1 \arrow[u] \arrow[ull] \\
 & \vdots &
\end{tikzcd}
$$

1 min

Example

- $s_1 s_0 \le s_1 s_0 s_1$

  since $(s_1 s_0) s_1 = s_1 s_0 s_1$
  & both reduced.

- $s_0 s_1 \le s_1 s_0 s_1$

  since $s_0 s_1 \cdot (s_1 s_0 s_1 s_0 s_1)$
  $= s_1 s_0 s_1$

  & $s_1 s_0 s_1 s_0 s_1$
  $= (s_1 s_0) s_1 (s_1 s_0)^{-1}$.

What's $P_{y,w}$ in $\widehat{sl_2}$?

For $s \in S$. $C_s = 1 + T_s$,

$C_{s_0 s_1} = 1 + T_{s_0} + T_{s_1} + T_{s_0} T_{s_1}$

$C_{s_1 s_0} = 1 + T_{s_1} + T_{s_0} + T_{s_1} T_{s_0}$

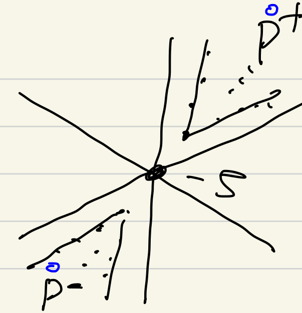
$C_{s_0 s_1 s_0} = 1 + T_{s_0} + T_{s_1} + T_{s_0 s_1} + T_{s_1 s_0} + T_{s_0 s_1 s_1}$

$C_{s_1 s_0 s_1} = \quad - - \cdot$

$$\begin{cases} P_{y,w} = 1 & \text{if } y \leq w \\ P_{y,w} = 0 & \text{otherwise} \end{cases}$$

Example: If $|S| = 2$

$$\begin{cases} P_{y,w} = 1 & \text{for any } y, w \in W \text{ with } y \leq w \\ P_{y,w} = 0 & y > w. \end{cases}$$



Recall: KL Conj:

when $\mathring{\mathfrak{g}}$ finite type

① For $\lambda \in \mathring{P}^-$ & $w \in \mathring{W}$

(a) $ch\, L(w \circ \lambda) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\, ch(M(y \circ \lambda))$

(a') $ch\, M(w \circ \lambda) = \sum_{y \leq w} P_{ww_0,\, yw_0}(1)\, ch(L(y \circ \lambda))$

② For $\lambda \in \mathring{P}^+$ & $w \in \mathring{W}$

(b) $ch\, L(w \circ \lambda) = \sum_{y \geq w} (-1)^{\ell(w)-\ell(y)} P_{yw_0,\, ww_0}(1)\, ch(M(y \circ \lambda))$

(b') $ch\, M(w \circ \lambda) = \sum_{y \geq w} P_{w,y}(1)\, ch(L(y \circ \lambda))$.

(a) $\Leftrightarrow$ (a') Since: $\displaystyle\sum_{y \le x \le w} (-1)^{\ell(x)-\ell(y)} P_{yx} P_{ww_0, xw_0} = \delta_{y,w}$

(b) $\Leftrightarrow$ (b')

Start from $\lambda \in P^+ \overset{w_0}{\rightsquigarrow}$ weights above the Berlin wall

$\lambda \in P^+ \overset{w_0}{\rightsquigarrow}$ weights below the Berlin wall.

λ lies on the Berlin Wall $\overset{w_0}{\rightsquigarrow}$ on the Berlin wall.

For affine Weyl gp W, w_0 doesn't exist.

$\Rightarrow$ Consider generalization of (a) & (b').

(integral, positive level non-crit.)

Thm A (Kashiwara–Tanisaki, 90, 91. Conj. by Deodhar–Gabber–Kac 82)
(Casian 90.

For any symmetrizable KM Lie alg $\mathfrak{g}$. $\forall \lambda \in P^+$, $w \in W$. We have:

$$(b'): \quad ch(M(w\circ\lambda)) = \sum_{y \geq w} P_{w,y}(1) \; ch\, L(y\circ\lambda).$$

Integral, negative non-crit level)

Thm B (Kashiwara–Tanisaki 95, Conj by Lusztig)
Casian

For any affine Lie alg $\mathfrak{g}$, $\forall \lambda \in P^-$, and $w \in W$,

$$(a). \quad ch(L(w\circ\lambda)) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \; ch(M(y\circ\lambda)).$$

Only for affine or of finite type! (any imaginary root β, $(\beta|\beta)=0$)

Note: Otherwise, the KL formula is not correct.

since: $\exists$ imaginary root. s.t $(\delta|\delta) < 0$

- The proof is similar to f. dim case of symmetrizable KM

$$n_k^{\pm} := \bigoplus_{\substack{\alpha\in\Delta^+ \\ ht(\alpha)\geq k}} \mathfrak{g}_{\mp\alpha} \qquad \left(\underline{\text{Recall}}:\ \alpha = \sum_{i=0}^{n} m_i \alpha_i,\quad ht(\alpha) = \sum_i m_i \right)$$

Define group scheme $H = \mathrm{Spec}\ \mathbb{C}[P]$ f. dim

$N^{\pm} :=$ the proj limit of $\underbrace{\exp(\underbrace{n^{\pm}/n_k^{\pm}}_{\text{f. dim nilpotent}})}_{\text{unipotent alg gr}}$

$B^{\pm} = H \ltimes N^{\pm}$.

G scheme (not a group scheme, but a scheme with a locally free left action of B^- & a locally free right action of B^+)

$X = G/B^+$.

Schubert Varieties $X^w := B^- w B^+/B^+$ $\quad w\in W$

$X_w := B^+ w B^+/B^+$.

<u>prop</u> (Kashiwara) ① $X = \coprod_{w \in W} X^w$

② $X^w \cong \mathbb{C}^\infty$ (unless $\mathfrak{g}$ is f. dim)

& $\operatorname{codim} X^w = \ell(w)$

③ $\overline{X^w} = \coprod_{y \geq w} X^y$.

<u>prop 2.6</u> (KT) ① $\coprod_{w \in W} X_w \subseteq X$ and equality holds if $\mathfrak{g}$ is f. dim

② $X_w \cong \mathbb{C}^{\ell(w)}$.

③ $\overline{X_w} = \coprod_{y \leq w} X_y$.

In <u>thm A</u>. Use left D-mods supported on $\overline{X^w}$.

<u>thm B</u> use right D-mods supported on $\overline{X_w}$.

Affine flag mfd is ∞-dim $\Rightarrow$

The cat left D-mod $\not\simeq$ The cat of right D-mods

not equivalent

behave well under inverse images

behave well under direct images

left D-mod = Inductive limit w.r.t a tower of f. dim varieties via inverse images

Right D-mod = proj limit w.r.t the same tower via direct images.

Rmk: 1) Rational, non-critical λ: Kashiwara-Tanisaki 96', of affine.

$\lambda \in \mathfrak{h}^*$ s.t $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ & $\langle \lambda+\rho, c \rangle \neq 0$.

$\Rightarrow \operatorname{char}(L(\lambda)) =$ KL polynomial

2) Critical case. Arakawa 07

$\lambda \in P^+_{crit} = \{ \lambda \in \mathfrak{h}^* \mid \mathring{\lambda} \in \mathring{P}^+ \ \& \ \langle \lambda, c \rangle = -h^\vee \}$

$\operatorname{char} L(\lambda) =$ explicit formula.