Gelfand Tsetlin algebra

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Gelfand-Tsetlin algebra

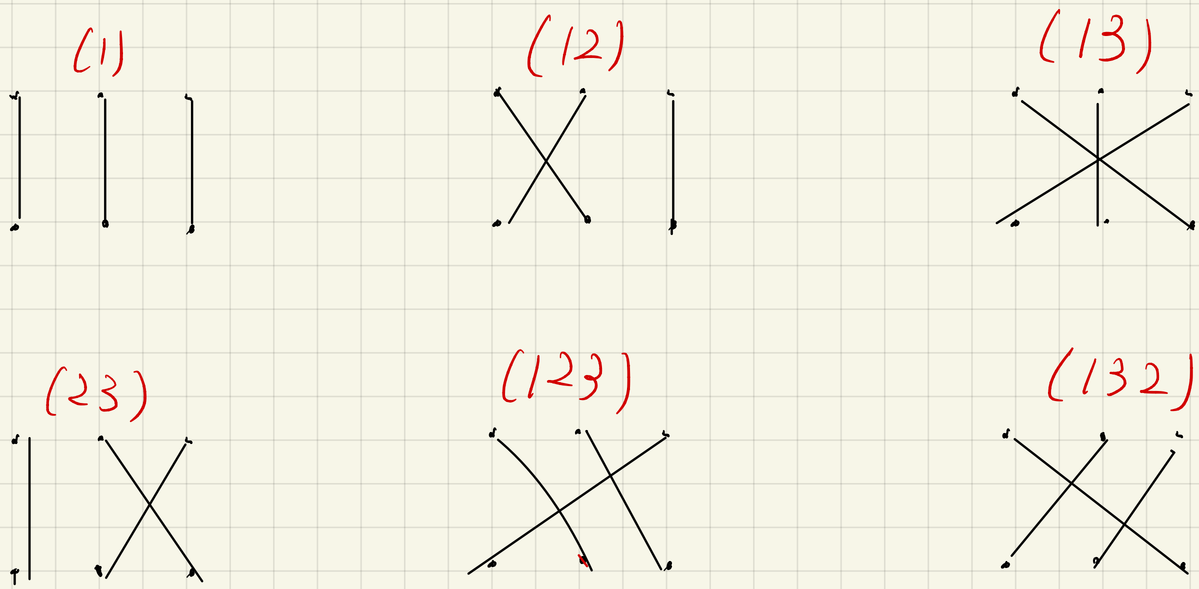
Symmetric group

$$S_n = \langle s_1 s_2 \cdots s_{n-1} \mid s_i^2 = 1 \quad s_i s_j = s_j s_i \text{ if } |i-j| > 1. \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$$

For example,

$$S_3 = \langle s_1, s_2 \rangle = \{ (1), (12), (13), (23), (123), (132) \}$$

Pictorially



Gelfand - Tsetlin algebra

Chain condition

finite group

Let

$$\{1\} = G(0) \subseteq G(1) \subseteq \cdots \subseteq G(n) = G.$$

Define

$$G(n)^{\wedge} = \{ \text{ equivalence class of irred rep}^{n} \text{ of } G(n) \}.$$

Example

$$\{1\} = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$$

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m).$

$$S_n^{\wedge} = \{ V^{\lambda} \mid \lambda \text{ is partition and } \lambda_1 + \lambda_2 + \cdots + \lambda_m = n \}$$

Gelfand-Tsetlin algebra

Let

$$\{1\} = G(0) \subseteq G(1) \subseteq \cdots \subseteq G(n) = G.$$

Define

$$G(n)^\wedge = \{ \text{equivalence class of irreducible rep}^{\underline{n}} \text{ of } G(n) \}.$$

Branching graph of $G(n)$

Vertices: $\bigcup_{n \geq 0} G(n)^\wedge$

Edges: $V^\lambda \overset{k}{\text{———}} V^\mu$ if $\dim\left(\mathrm{Hom}_{G(m-1)}(V^\mu, V^\lambda)\right) = k.$

V^λ: $G(m)$-module

V^μ: $G(m-1)$-module

Branching graph of $G(n)$

Vertices: $\bigcup_{n \geq 0} G(n)^{\wedge}$

Edges: $V^{\lambda} \overset{k \text{ edges}}{\text{———}} V^{\mu}$ if $\dim\left(\mathrm{Hom}_{G(m-1)}(V^{\mu}, V^{\lambda})\right) = k$.

$V^{\lambda} \in G(m)$-module, $\quad V^{\mu} \in G(m-1)$-module

Example Branching graph of S_n

Vertices: $\{ V^{\lambda} \mid \lambda \text{ is a partition with } \max(|\lambda|) \leq n \}$.

Edges: $\dim\left(\mathrm{Hom}_{S_{m-1}}(V^{\mu}, V^{\lambda})\right) \in \{0,1\}$. $\longleftarrow$ simple branching

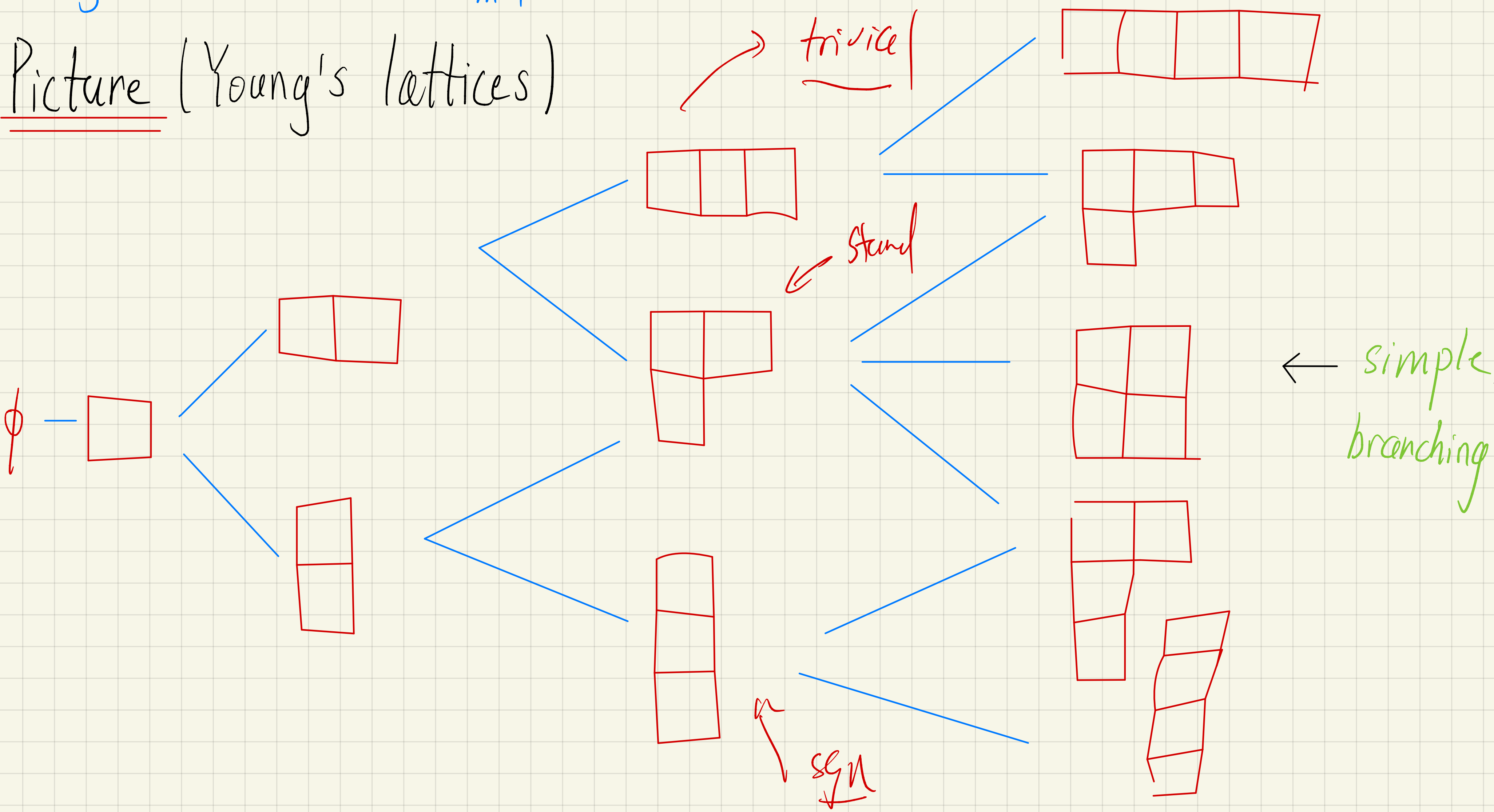
$V^{\mu} \in S_{m-1}$-module, $\quad V^{\lambda} \in S_m$-module.

Example Branching graph of S_n

Vertices: $\{ V^\lambda \mid \lambda$ is a partition with $\max |\lambda| = n \}$.

Edges: $\dim(\mathrm{Hom}_{S_{m-1}}(V^\lambda, V^\mu)) \in \{0,1\}$. $\mu \subseteq \lambda$.

Picture (Young's lattices)



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a is real then $a^* = a$.

Main point

finite dim $*$-algebra

Lemma 1

(If $a \in A$ and a is real then $a^* = a$.) then A is commutative.

Prop 1

$Z(M, N)$ is commutative iff $\dim(\mathrm{Hom}_N(V^\mu, V^\lambda)) \in \{0, 1\}$.

$N \subseteq M$

simple branching

Lemma 2

If $g \in S_n$, then there exists $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

Th$^{\underline{m}}$ 1

group algebra

The branching graph of $\mathbb{C}[S_n]$ is simple.

Gelfand - Tsetlin algebra

Lemma I

If ~~for~~ all real $a \in A$ it is true $a^* = a$ then A is commutative.

Involutive algebra ($*$-algebra).

$*$-algebra A is an algebra with $*: A \longrightarrow A$.
$a \longmapsto a^*$.

① $(a^*)^* = a$ (second order).

② $(ab)^* = b^* a^*$ (anti-automorphism)

③ $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$, $\alpha, \beta \in \mathbb{C}$.

complex conjugation.

Involutive algebra ($*$-algebra).

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① $((a)^*)^* = a$ (second order)

② $(ab)^* = b^* a^*$ (anti-automorphism)

③ $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$.

Example $\mathbb{C}[S_n]$ is a finite dimensional $*$-algebra

$$* : \mathbb{C}[S_n] \longrightarrow \mathbb{C}[S_n]$$

$$\sum_{g \in S_n} c_g g \longmapsto \sum_{g \in S_n} \overline{c_g}\, g^{-1}$$

$Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$

$*$-algebra

Gelfand - Tsetlin algebra

$a = \sum_{g\in G} c_g\, g \longrightarrow c_g \in \mathbb{R}$.

(Observation) $\longleftarrow$ $*$-algebra finite-dim group algebra.

If $a,b \in A$ and $a^* = a$, $b^* = b$ and a,b are real, then $\big((ab)^* = ab$ and ab is real$\big)$

Proof: $\Big(\big(\underbrace{\sum_{g\in G} c_g\, g}_{\overset{\shortparallel}{a}}\big)\big(\underbrace{\sum_{g'\in G} c_{g'}\, g'}_{\overset{\shortparallel}{b}}\big)\Big)^*$

$$= \Big(\sum_{g\in G}\sum_{g'\in G} c_g c_{g'}\, g g'\Big)^*$$

$$= \sum_{g\in G}\sum_{g'\in G} \overline{c_g}\,\overline{c_{g'}}\, g'^*\, g^*$$

$$= \sum_{g\in G}\sum_{g'\in G} c_g c_{g'}\, g'^*\, g^*$$

$$= \Big(\sum_{g'\in G} c_{g'}\, g'^*\Big)\Big(\sum_{g\in G} c_g\, g^*\Big) = \Big(\overset{a}{\overset{\shortparallel}{\sum_{g\in G} c_g\, g}}\Big)\Big(\overset{b}{\overset{\shortparallel}{\sum_{g'\in G} c_{g'}\, g'}}\Big)$$

$$a = \Big(\sum_{g\in G} c_g\, g\Big)^* = \sum_{g\in G} c_g\, g$$

$$\| $$

$$\sum_{g\in G} c_g\, g^* \Longrightarrow c_g = c_{g^*} \text{ and } c_{g'} = c_{g'^*}$$

Lemma 1.

$\mathbb{C}$-algebra with an $*$ operator.

If $x \in A$ and x is real then $x^* = x$ $\Longrightarrow$ A is commutative.

Proof: By observation, let $a, b \in A$.

$(a+bi)(c+di)$

$$a \cdot b = (ab)^* = b^* a^* = ba \qquad = (c+di)(a+bi)$$

observation anti-auto self-adjoint.

Hence, A is commutative.

Gelfand - Tsetlin algebra

$\boxed{\text{~~Main point~~ Lemma 1.}}$

If ~~for~~ all real $a \in A$ it is true $a^* = a$ then A is commutative.

Lemma 1

Prop 1

$Z(M,N)$ is commutative iff $\dim(\mathrm{Hom}_N(V^\lambda, V^\mu)) \in \{0,1\}$.

Simple branching

Lemma 2

If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

Th$^{\underline{m}}$ 1

The branching graph of $\mathbb{C}[S_n]$ is simple.

Gelfand - Tsetlin algebra

Let M, N be semisimple finite-dim $\mathbb{C}$-algebra and $N \subseteq M$.

Define

$$Z(M,N) = \{ m \in M \mid m \cdot n = n \cdot m \text{ for all } n \in N \}.$$

Example For $\mathbb{C}[S_n]$, ← group algebra

$$Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$$

conjugates

$$= \mathbb{C}\text{-span } Z(\mathbb{C}[S_{n-1}]) \cup \left(\sum_e (i_1^{(1)}, i_2^{(2)}, \dots, i_k^{(k)}, n) \dots \right.$$

$$(i_1^{(1)}, i_2^{(2)}, \dots, i_l^{(l)})$$

?!

Example For $\mathbb{C}[S_n]$,

$h \in S_{n-1}$ $\boxed{hgh^{-1}}$

$$Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]) = \mathbb{C}\text{-span } Z(\mathbb{C}[S_{n-1}]) \cup$$

$$\left\{ \sum_{e} (i_1^{(1)}, i_2^{(2)}, \ldots, i_k^{(k)}, n) \cdots (i_1^{(1)}, i_2^{(2)}, \ldots, i_l^{(l)}), \ldots \right\} \rightarrow \mathbb{C}[S_n].$$

Let $x \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$. Then

$$x = \sum_{g \in S_n} c_g g = \underbrace{\sum_{g \in S_{n-1}} c_g g}_{x_1} + \underbrace{\sum_{\substack{g \in S_n \\ g \notin S_{n-1}}} c_g g}_{x_2}$$

$h \in S_{n-1}$

and $hxh^{-1} = x \implies x_1 \in Z(\mathbb{C}[S_{n-1}])$

$\implies x_2 \in x - x_1 \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$.

Let $x \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$. Then

$$x = \sum_{g \in S_n} c_g g = \underbrace{\sum_{g \in S_{n-1}} c_g g}_{x_1} + \underbrace{\sum_{\substack{g \in S_n \\ g \notin S_{n-1}}} c_g g}_{x_2}$$

① $x_1 \in Z(S_{n-1})$

② $x_2 \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$.

③ $g = (\cdots n) \cdots (\cdots)$. if $g \in X_2$.

Since $hxh^{-1} = x$ for $h \in S_{n-1}$, then

$$c_g = c_{hgh^{-1}} \quad \text{for} \quad h \in S_{n-1}$$

and

$$x_2 = \sum_{g \in G} c_g g = \sum_{g \in R_{S_n}} c_g (g + h_1 g h_1^{-1} + h_2 g h_2^{-1} + \cdots)$$

Gelfand - Tsetlin algebra

① $X_1 \in Z(S_{n-1})$

② $X_2 = \sum_{\substack{g \in S_n \\ g \notin S_{n-1}}} c_g g = \sum c_g (g + h_1 g h_1^{-1} + h_2 g h_2^{-1} \cdots)$

③ $X = X_1 + X_2 \in Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$.

Example For $\mathbb{C}[S_n]$,

$$Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]) = \mathbb{C}\text{-span } Z(\mathbb{C}[S_{n-1}]) \cup$$

$$\left\{ \sum_{e} (i_1^{(1)}, i_2^{(2)}, \ldots, i_k^{(k)}, n) \cdots (i_1^{(1)}, i_2^{(2)}, \ldots, i_l^{(l)}), \ldots \right\}$$

Gelfand - Tsetlin algebra

Main point

Lemma 1

If $x \in A$ and x is real and $x^* = x$ then

A is commutative.

Prop 1

$Z(M,N)$ is commutative $\boxed{\text{iff}}$ $\dim(\mathrm{Hom}_N(V^{\mu}, V^{\lambda})) \in \{0,1\}$.

simple branching

Lemma 2

If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

Th^m 1

The branching graph of $\mathbb{C}[S_n]$ is simple.

Gelfand-Tsetlin algebra

algebra

Lemma Let V^λ be an M-representation
V^μ be an N-representation. $N \subseteq M$.

Then $\mathrm{Hom}_N(V^\mu, V^\lambda)$ is irreducible $Z(M,N)$-rep^n.

Proof: (Ideal)

Step 1. Consider

$$\begin{tikzcd}
N \times Z(M,N) \arrow[r, "f"] \arrow[d] & M \\
N \otimes_{\mathbb{C}} Z(M,N) \arrow[ur, dashed, "\exists!\, \tilde{f}"] \arrow[d] & \\
N \otimes_{N \cap Z(M,N)} Z(M,N) \arrow[uur, dashed, "\tilde{f}_2"'] &
\end{tikzcd}$$

If $a \in N \cap Z(M,N)$ then

$\tilde{f}_2(m \otimes an - am \otimes n)$

$= m \cdot an - am \cdot n$

$= 0$

<u>Proof</u>: (Ideal)

Step 1. Consider

$$\begin{tikzcd}
N \times Z(N,M) \arrow[r, "f"] \arrow[d] & M \\
N \otimes Z(N,M) \arrow[ur, dashed, "\exists! \tilde{f}_1"] \arrow[d] & \\
N \otimes_{\mathbb{C}} Z(N,M) / N \cap Z(N,M) \arrow[uur, dashed, "\tilde{f}_2"'] &
\end{tikzcd}$$

If $a \in N \cap Z(N,M)$ then

$$\tilde{f}_2(n \otimes am - an \otimes m) = na \cdot m - an \cdot m = 0$$

$N \otimes_{N \cap Z(M,N)} Z(M,N) \hookrightarrow M.$

$\subseteq$

Step 2. Note

$$\operatorname{Res}^M_N V^\lambda \cong \bigoplus_\mu G_\mu V^\mu \cong \bigoplus_\mu \operatorname{Hom}_N(V^\mu, V^\lambda) \otimes V^\mu$$

$\oplus h_\mu.$

$Z(M,N)$-module

N-module

$$f(v) \longleftarrow f \otimes v$$

Step 3

$$\mathrm{Hom}_{\substack{N\otimes Z\\ N\cap Z}}\left(\mathrm{Hom}_N(V^\mu,V^\lambda)\otimes V^\lambda,\ \mathrm{Res}^{M}_{\substack{N\otimes Z\\ N\cap Z}} V^\lambda\right) \simeq$$

$$\mathrm{Hom}_M\left(\mathrm{Ind}^{M}_{\substack{N\otimes Z\\ N\cap Z}}\left(\mathrm{Hom}_N(V^\mu,V^\lambda)\otimes V^\lambda\right),\ V^\lambda\right).$$

$$h_\mu \longmapsto \tilde{h}_\mu.$$

Note

$$\tilde{h}_\mu : \mathrm{Ind}^{M}_{\substack{N\otimes Z\\ N\cap Z}}\big(\underbrace{\mathrm{Hom}(V^\mu,V^\lambda)\otimes V^\lambda}_{=:\,H}\big) \longrightarrow V^\lambda.$$

Step 4

Assume H is not irreducible.

$$0 \longrightarrow \underset{\neq 0}{H_1} \xrightarrow{\iota} H \longrightarrow \underset{\neq 0}{H_2} \longrightarrow 0.$$

H_2: coker ι.

Step 4 Assume H is not irreducible.

$$\begin{tikzcd} 0 \arrow[r] & H_1 \arrow[r, "\iota"] \arrow[d] & H \arrow[r] & H_2 \arrow[r] & 0 \\ 0 \arrow[r] & H_1 \otimes V^\mu \arrow[r, "\iota"] \arrow[dr, "h_\mu \circ \iota"'] & H \otimes V^\mu \arrow[r] \arrow[d, "h_\mu"] & H_2 \otimes V^\mu \arrow[r] & 0 \\ & & \mathrm{Res}^M_{N\otimes Z} V^\lambda & & \end{tikzcd}$$

$H_1 \neq 0$, $H_2 \neq 0$, $H_2 = \operatorname{coker} \iota$.

irreducible.

$H_1 \subsetneq H$.

Last step:

$$\mathrm{Ind}^M_{N\times Z}(H_1 \otimes V^\mu) \xrightarrow{h_\mu \circ \iota} V^\lambda \quad \xRightarrow{\mathrm{Hom}(-,V^\mu)} \quad H_1 \twoheadrightarrow H$$

$\lightning$

Gelfand - Tsetlin algebra

Let M, N be semisimple finite dim $\mathbb{C}$-algebra and $N \subseteq M$.

Define

$$Z(M,N) = \{ m \in M \mid m \cdot n = n \cdot m \text{ for all } n \in N \}.$$

Prop 1

branching simple.

$Z(M,N)$ is commutative $\iff$ $\dim(\mathrm{Hom}(V^\lambda, V^\mu)) \in \{0,1\}$.

Proof: $(\Rightarrow)$ $\boxed{(\Leftarrow)}$

Lemma $\Longrightarrow$ $\mathrm{Hom}_N(V^\mu, V^\lambda)$ is irreducible $Z(M,N)$-rep.

$Z(M,N)$ is commutative $\longrightarrow$ $\Longrightarrow$ $\dim(\mathrm{Hom}(V^\mu, V^\lambda)) = \{0,1\}$.

Gelfand - Tsetlin algebra

Main point

Lemma 1: If $x \in A$ and x is real and $x^* = x$ then A is commutative.

Prop 1: $Z(M,N)$ is commutative iff $\dim(\mathrm{Hom}_N(V^\lambda, V^\mu)) \in \{0,1\}$.

simple branching

Lemma 2: If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

Th^m 1

The branching graph of $\mathbb{C}[S_n]$ is simple.

Gelfand - Tsetlin algebra

Lemma 1

there exist

If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

$(1, \ldots, 19)$

S_{18}.

Proof: (Example)

Let

$$g = (19,5,2,3)(1,6,8,15,12)(18,10,4,17,16,7)(14,11,9,1)(13) \in S_{19}$$

$$g^{-1} = (19,3,2,5)(1,12,6,8,15)(18,7,16,17,4,10)(14,1,9,11)(13)$$

Now

$$h = \begin{pmatrix} 19 & 5 & 2 & 3 & 1 & 6 & 8 & 15 & 12 & 18 & 10 & 4 & 17 & 16 & 7 & 14 & 11 & 9 & 1 & 13 \\ 19 & 3 & 2 & 5 & 1 & 12 & 6 & 8 & 15 & 18 & 7 & 16 & 17 & 4 & 10 & 14 & 1 & 9 & 11 & 13 \end{pmatrix}$$

Gelfand - Tsetlin algebra

$\boxed{\text{Main point}}$

Lemma 1

If $x \in A$ and x is real and $x^* = x$ then ✓

A is ~~commutative~~ commutative. ✓

Prop 1

$Z(M,N)$ is commutative iff

$\dim(\mathrm{Hom}(V^\lambda, V^\mu)) \in \{0,1\}$

simple branching

Lemma 2

If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$. ✓

$\mathrm{Th}^{\underline{m}}$ 1

$\longleftarrow$

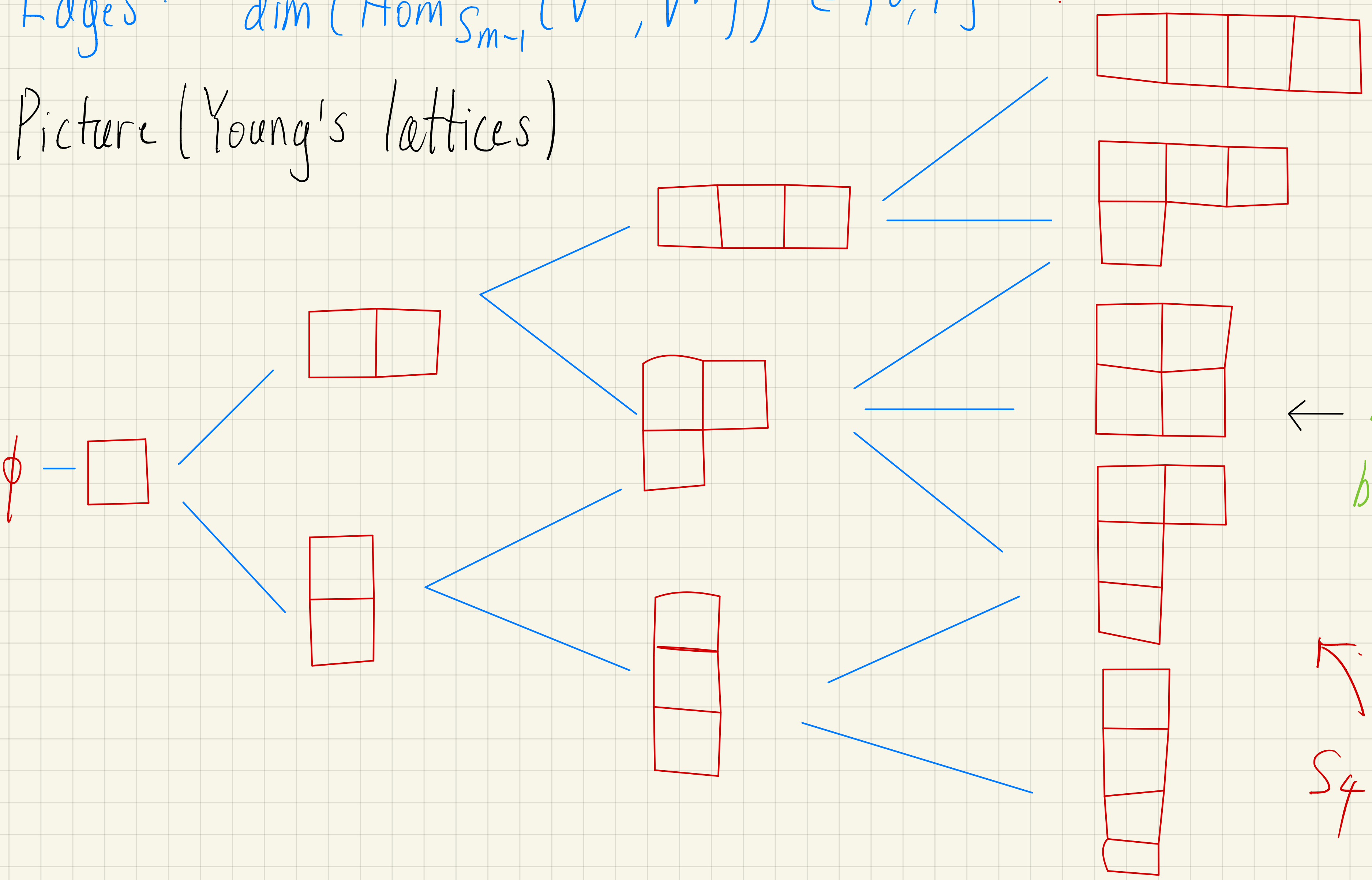
The branching graph of $\mathbb{C}[S_n]$ is simple.

Example Branching graph of S_n

Vertices : $\{ V^\lambda \mid \lambda$ is a partition with $\underline{\max |\lambda| = n}\}$.

Edges : $\dim(\mathrm{Hom}_{S_{m-1}}(V^\lambda, V^\mu)) \in \{0,1\}$

Picture (Young's lattices)



simple branching

S_4

Example $\mathbb{C}[S_n]$ is a finite dimensional $*$-algebra

$$* : \mathbb{C}[S_n] \longrightarrow \mathbb{C}[S_n]$$

$$\sum_{g\in S_n} \longmapsto \sum_{g\in S_n}$$

Lemma 3

If $g \in S_n$, then $h \in S_{n-1}$ s.t. $hgh^{-1} = g^{-1}$.

Observation If $\in Z(\quad , \quad)$, then

$$a = \sum_{g\in S_n} \quad = \quad + \sum_{\substack{g\in S_n \\ g\neq g'}}$$

By , there exists $\in S_{n-1}$,

$$\sum_{g\in S_n} \quad = a = hah^{-1} = \quad + \sum_{\substack{g\in S_n \\ g\neq g'}}$$

$\uparrow$

Hence

$=$

Galfand - Testlin algebra

Th$^{\underline{m}}$ 1

The branching graph of $\mathbb{C}[S_n]$ is simple.

Proof: $\overset{\text{Prop 1}}{\Longleftrightarrow}$ $Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$

$a \in$ and a is $\overset{\text{Lemma 1}}{\Longrightarrow}$ $Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$

then $=$

Then

$$a^* = \Big(\sum_{g \in G} \Big)^* = \sum_{g \in G} = \sum_{g \in G} = \sum_{g \in G}$$

$$= \sum_{g \in G} =$$

Next talk:

1. Gelfand-Tsetlin algebra (diagonal)

2. Young - Jucys - Murphy elements

3. Gelfrand Testlin algebra is generated by YJM elements.