$\mathcal{A} \subseteq \mathbb{Z}[[\xi_1, \xi_2, \ldots]]$

PSH structure on $\mathcal{A}$ in terms of ξ_i.

$$R \longrightarrow \mathcal{A}_R \longrightarrow \mathcal{A}$$

explicitly write down comultiplication in $\mathcal{A}$.

$\zeta = \{\zeta_1, \zeta_2, \ldots\}$ and $\eta = \{\eta_1, \eta_2, \ldots\}$

$\mathcal{A}\otimes\mathcal{A} \hookrightarrow \mathbb{Z}[[\eta, \zeta]]$

$F \otimes G \longmapsto F(\eta)\cdot G(\zeta)$

$\mathcal{A}\otimes\mathcal{A}$ is realized as a subalgebra in $\mathbb{Z}[[\eta,\zeta]]$ consisting of those series that are symm separately wrt η, ζ.

Prop: $m^* : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ has the form

$$\underbrace{\mathcal{A} \xrightarrow{c} \mathcal{A}(\eta,\zeta) \hookrightarrow}_{m^*} \mathcal{A}\otimes\mathcal{A} \subseteq \mathbb{Z}[[\eta,\zeta]].$$

Pf: Just have to verify m^* on generators of $\mathcal{A}$.

$z_n \mapsto \sum_i \xi_i^n$. $\cancel{\mathcal{A}_{\mathbb{Q}}}$

z_n are gens for $\mathcal{A}_{\mathbb{Q}}$ but this doesn't matter since no torsion.

$m^*(z_n) = z_n \otimes 1 + 1 \otimes z_n$

$c(\xi_1^n + \xi_2^n + \cdots) = (\sum_i \eta_i^n) + (\sum_i \zeta_i^n)$

$\updownarrow$

$z_n \otimes 1 + 1 \otimes z_n$ $\square$

Inner product.

FACT from lin Alg: An inner product on a fin dim vector space V can be characterized by a nondegenerate, symmetric tensor

$$g \in V\otimes V.$$

$\langle , \rangle$ on V, $\{e_1, \ldots, e_p\}$, $\{e_1^\perp, \ldots, e_p^\perp\}$ wrt $\langle , \rangle$,

$$g = \sum_{i=1}^p e_i \otimes e_i^\perp \in V\otimes V$$

$$V^* \xrightarrow{\sim} V$$

$g = \sum e_i \otimes e_i^{+}$

explicitly describe the tensor $g_n \in \mathcal{A}_n \otimes \mathcal{A}_n$ corresponding to the inner product on $\mathcal{A}_n$.

($R_n \cong$)

Prop: If $\mathcal{A} \otimes \mathcal{A}$ is realized as subring of $\mathbb{Z}[[\eta, \xi]]$. Then g_n is equal to the sum of all monomials of degree n in indeterminates

$$\eta \cdot \xi = \{\eta_i \xi_j \mid i, j = 1, 2, \dots\}.$$

Pf: our basis is set of x_λs with $|\lambda| = n$.

$$P_- : R \longrightarrow \mathcal{A}_R \xrightarrow{\sim} R \otimes \mathcal{A}$$

$$\begin{array}{c} \text{\rotatebox{90}{$\cong$}} \\ \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \quad \text{since } R \cong \mathcal{A}. \end{array}$$

$x_n \in \mathcal{A}$, wtp $g_n = x_n(\eta \cdot \xi)$

$$P(x_n) = \sum_{|\lambda| = n} x_\lambda \otimes u_\lambda = g_n$$

($u_\lambda = x_\lambda^{\perp}$)

EX

FACT: If $\mathcal{A} \otimes \mathcal{A} \subseteq \mathbb{Z}[[\eta, \xi]]$. Then $P : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ has

$$P(F)(\eta, \xi) = F(\eta \cdot \xi).$$

So $P(x_n)(\eta, \xi) = \sum_{|\lambda| = n} x_\lambda(\eta) \cdot u_\lambda(\xi)$

by fact above, $= x_n(\eta \cdot \xi)$

$$x_n(\eta \cdot \xi) = g_n \qquad \square$$

THE DUAL $\mathcal{A}^*$.

$\mathcal{A}^* = \bigoplus_{n \geq 0} \mathcal{A}_n^*$ where $\mathcal{A}_n^*$ is dual of $\mathcal{A}_n$.

Some facts:

- $\mathcal{A}^*$ has a Hopf algebra structure.
- The inner product $\langle , \rangle$ on $\mathcal{A}$ gives rise to the iso $g : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$
- The self-adjointness axiom on $\mathcal{A}$ means that g is an iso of Hopf algebras.

$B_{\mathbb{Q}} = \mathbb{Q}[[\xi_1, \xi_2, \ldots]]$

Let B^* to be the algebra of differential operators on $B_{\mathbb{Q}}$.

degree 1.

gen by $\frac{\partial}{\partial \xi_i} : B_{\mathbb{Q}} \longrightarrow B_{\mathbb{Q}}$.

B^* is graded. ~~deg~~

The group S_∞ acts on B^* via
$\sigma(D) = \sigma \circ D \circ \sigma^{-1}$, $D \in B^*$

B^*/S is the quotient of B^* by the subspace spanned by
$\sigma D - D$.

B^*/S is a graded vector vector space over $\mathbb{Q}$.

but not an algebra.

For any $\alpha = (a_1, a_2, \ldots)$ we have

$$D_\alpha = \frac{1}{a_1! a_2! \cdots} \left(\frac{\partial}{\partial \xi_1}\right)^{a_1} \circ \left(\frac{\partial}{\partial \xi_2}\right)^{a_2} \circ \cdots$$

$\lambda \in \mathcal{P}_n$ then D_λ form a basis for $(B^*/S)_n$.

$D \in (B^*/S)_n \leftrightarrow$ functional
$(A_{\mathbb{Q}})_n \longrightarrow \mathbb{Q}$

<u>Prop</u>: The graded group A^* is naturally identified with the lattice in B^*/S spanned by elements D_λ.

$$g : A \xrightarrow{\sim} A^*, \quad g(x_\lambda) = D_\lambda$$
$\lambda \in \mathcal{P}$.

Section 5 Notes (P12)

Wednesday, 31 March 2021 12:34 PM

η, ζ.

$B^* \otimes B^*$ with $B^*(\eta, \zeta)$ via

$D_1 \otimes D_2 \longmapsto D_1(\eta) \circ D_2(\zeta)$.

$A^* \otimes A^*$ is identified with the lattice in $B^*(\eta,\zeta) / S_\infty(\eta) \times S_\infty(\zeta)$, spanned by the operators $D_\lambda(\eta) \circ D_\mu(\zeta)$ for $\lambda, \mu \in \mathcal{P}$.

Proposition. (a) The multiplication $\mathcal{A}^*\otimes\mathcal{A}^*\longrightarrow\mathcal{A}^*$ is induced by an isomorphism $\mathcal{B}^*(\eta,\zeta)\xrightarrow{\sim}\mathcal{B}^*$ obtained by means of an (arbitrary) bijection $\{\eta_1,\eta_2,\dots,\zeta_1,\zeta_2,\dots\}\xrightarrow{\sim}\{\xi_1,\xi_2,\dots\}$.

(b) The comultiplication $\mathcal{A}^*\longrightarrow\mathcal{A}^*\otimes\mathcal{A}^*$ is induced by the ring homomorphism $\mathcal{B}^*\longrightarrow\mathcal{B}^*(\eta,\zeta)$ (with respect to composition as a multiplication), sending $\frac{\partial}{\partial\xi_i}\longmapsto\frac{\partial}{\partial\eta_i}+\frac{\partial}{\partial\zeta_i}$ (i=1,2,...).

Proposition. Let $v\in\mathcal{A}$ and $g(v)=D\in\mathcal{A}^*$ (see 5.8).

Then for all $F\in\mathcal{A}(\xi)$

$$v^*(F)\,(\xi)=D(\eta)\left[F(\xi,\eta)\right]\Big|_{\eta=0}$$

(this means that we must write down the differential operator D in indeterminates η, evaluate it at the polynomial $F(\xi,\eta)$ and then put $\eta_i=0$ (i = 1,2,...).

Cor: $x^*:\mathcal{A}\to\mathcal{A}$ acts as follows

$$\mathcal{A}(\xi_1,\xi_2,\dots)\xrightarrow{C}\mathcal{A}(\xi_0,\xi_1,\xi_2,\dots)\xrightarrow{\xi_0\mapsto 1}\mathcal{A}(\xi_1,\xi_2,\dots)$$

$$R\xrightarrow{\sim}\mathcal{A}\xrightarrow[g]{\sim}\mathcal{A}^*$$

x^*

S-functions $\{\lambda\}$

$$\begin{matrix}\mathcal{P} & & R(S) & & \mathcal{A}\\ \lambda & \longleftrightarrow & \{\lambda\} & \longrightarrow & P(\{\lambda\})\end{matrix}$$

Fix $N\geq 1$

$$\Delta(\xi_1,\dots,\xi_N)=\prod_{1\leq i<j\leq N}(\xi_i-\xi_j)$$

Δ is an alternating polynomial in $\xi_1,\dots,\xi_N$.
Ω^N_n is the group of all integral alternating polynomials in $\xi_1,\dots,\xi_N$ of deg $n+C^2_N$

$\Delta \in \mathcal{A}$

$\Delta : \mathcal{A}_n^N \longrightarrow \Lambda_n^N$

$\lambda \in \mathcal{P}_n \quad r(\lambda) \leq N$ and $(\lambda) = (l_1, \ldots, l_N)$,

$\Delta_\lambda(\xi_1, \ldots, \xi_N) = \cancel{\left|\xi_1^{l_1+N-1} \quad \xi_2^{l_2+N-2} \quad \cdots \quad \xi_N^{l_N}\right|}$

$= \det\left(\xi_i^{l_j+N-j}\right)_{i,j=1,\ldots,N}$

Δ_λ / Δ form a $\mathbb{Z}$-basis in $\mathcal{A}_n^N$.

<u>Prop</u>: $\varphi_N : \mathcal{A} \longrightarrow \mathcal{A}^N$.

$\varphi_N(\{\lambda\}) = \Delta_\lambda(\xi_1, \ldots, \xi_N) / \Delta(\xi_1, \ldots, \xi_N)$.