Thm: Let R be a PSH-algebra with unique irreducible primitive element ρ of degree 1.

1). There exist unique irred. elements (x_n), (y_n) in R_n, $n \in \mathbb{Z}_{\geq 0}$ such that

- $\sum_{k=0}^{n} (-1)^k x_k \cdot y_{n-k} = 0 \quad (n \geq 1)$
- $R = \mathbb{Z}[x_1, x_2, \cdots]$
- $R = \mathbb{Z}[y_1, y_2, \cdots]$.

2). R has a unique non-trivial automorphism t such that $t(x_n) = y_n$, $t(y_n) = x_n$, $n \geq 1$

3). Any PSH-algebra R' with unique irr. primitive element ρ' of degree 1 is isomorphic to R as PSH-algebra. There are exactly two PSH-algebra isomorphism from R to R'.

- ρ^2 is a sum of two distinct irr. elements:

$$\rho^2 = x_2 + y_2$$

- $\forall\ n \in \mathbb{Z}_{\geq 0}$, $\exists!$ irr. element x_n, y_n such that $y_2^*(x_n) = 0$, $x_2^*(y_n) = 0$

$$\rho^*(x_n) = x_{n-1}, \quad \rho^*(y_n) = y_{n-1}$$

$\begin{cases} y_n \\ x_n \end{cases}$ is constructed inductively as the irr constituent of $\begin{cases} \rho \cdot y_{n-1} \\ \rho \cdot x_{n-1} \end{cases}$ which satisfies

$$\begin{cases} \rho^*(x_n) = x_{n-1} \\ \rho^*(y_n) = y_{n-1} \end{cases}$$

- If $0 \leq k \leq n$, then

$$x_k^*(x_n) = x_{n-k}, \qquad y_k^*(y_n) = y_{n-k}.$$

If $w \in \Omega$ is distinct from

$$x_0, \ldots, x_n$$

then $w^*(x_n) = 0$.

If $w \in \Omega$ is distinct from

$$y_0, \ldots, y_n$$

then $w^*(y_n) = 0$.

- For $n \geq 1$,

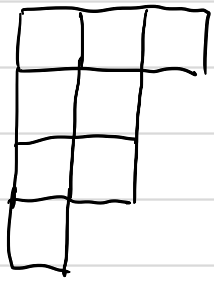
$$m^*(x_n) = \sum_{k=0}^{n} x_k \otimes x_{n-k}$$

$$m^*(y_n) = \sum_{k=0}^{n} y_k \otimes y_{n-k}$$

- This is a 1-to-1 correspondence

$$\mathcal{P} \longrightarrow \text{Set of all diagrams.}$$

e.g. $(3,2,2,1) \longmapsto$



$(0) \longmapsto \phi$

Goal: $R = \mathbb{Z}[x_1, x_2, \cdots]$

λ : partition, $\lambda = (l_1, l_2, \cdots, l_r)$

$x_\lambda = x_{l_1} \cdot x_{l_2} \cdots x_{l_r}$

$(x_\lambda \mid \lambda \in \mathcal{P}_n)$ forms a basis of R_n

e.g. $n = 3$, $((3), (2,1), (1,1,1))$

$$x_{(3)} = \underline{x_3}$$

$$x_{(2,1)} = \underline{x_2 \cdot x_1}$$

$$x_{(1,1,1)} = \underline{x_1 \cdot x_1 \cdot x_1}$$

<u>lemma</u> : $e_1, \cdots, e_p$: elements of a T-group

$(e_1, \cdots, e_p)$ is a basis of certain T-subgroup

iff $\det(\langle e_i, e_j \rangle)_{i,j=1,\cdots,p} = 1$

We have $(\chi_\lambda \mid \lambda \in \mathcal{P}_n)$.

Let $\lambda_1, \cdots, \lambda_p$ be all elements of $\mathcal{P}_n$ ordered in such a way that

for $i<j$, c.f (λ_j^t) is lexicographically higer than c.f (λ_i^t).

$\lambda_1 = (n), \quad \cdots\cdots, \quad \lambda_p = (1, 1, \cdots, 1)$.

<u>Claim</u> : $\det(\langle \chi_{\lambda_i}, \chi_{\lambda_j} \rangle) = 1$

Let $w \in R_n$ be an irr. element.

$$\langle \chi_{\lambda_p}, w \rangle = \langle \chi_{(1,\cdots,1)}, w \rangle$$
$$= \langle \chi_1^{\,n}, w \rangle$$
$$= \langle \rho^n, w \rangle > 0$$

$\Rightarrow$ $(\chi_{\lambda_i} \mid 1 \le i \le p)$ is a basis of R_n.

$\Rightarrow$ $R = \mathbb{Z}[x_1, x_2, \cdots]$.

Similarly,

$$R = \mathbb{Z}[y_1, y_2, \cdots,]$$

⌜ Conjugation :

Thm : A : connected Hopf algebra $\mathbb{K}$

a). $\exists !$ morphism of graded $\mathbb{K}$-modules

$$T : A \longrightarrow A \quad \text{s.t}$$

$$\begin{tikzcd} A \arrow[r, "m^*"] \arrow[drr, bend right, "e^*"'] & A \otimes A \arrow[r, "id \otimes T"] & A \otimes A \arrow[r, "m"] & A \\ & & \mathbb{K} \arrow[ur, bend right, "e"'] & \end{tikzcd}$$

b). If A has commutative multiplication comultiplication, then T is an involutive Hopf algebra automorphism of A.

⌟

Define $t : R \longrightarrow R$ in the following way

$$t\big|_{R_n} = (-1)^n \, T\big|_{R_n}$$

Claim: a). t is a positive involutive automorphism of the PSH-algebra R.

b). For all $n \in \mathbb{Z}$, $t(x_n) = y_n$

$t(y_n) = x_n$

Outline of proof:

- t is an isometry

$\Rightarrow$ t: PSH-algebra automorphism

- $m \circ (id \otimes T) \circ m^*(x_2) = 0$

$0 = T(x_2) - \rho^2 + x_2$

$\Rightarrow$ $t(x_2) = (-1)^2 T(x_2) = \rho^2 - x_2 = y_2$

- t isometry $\Rightarrow$ $(t(a))^* = t \circ a^* \circ t^{-1}$

$a \in R$

$\Rightarrow$ $(x_2^* \circ t)(x_n) = (t \circ y_2^*)(x_n) = 0$

$\Rightarrow$ $t(x_n) = y_n$

- ρ^n is the regular repn of S_n:

  ρ is the trivial repn of S_1.

  $\rho^n = \mathrm{Ind}_{S_1 \times \cdots \times S_1}^{S_n} \rho \otimes \cdots \otimes \rho$, $\rho \otimes \cdots \otimes \rho$ is the trivial repn of $S_1 \times \cdots \times S_1 \cong \{e\} \subset S_n$

  So $\mathrm{Ind}_{S_1 \times \cdots \times S_1}^{S_n} \rho \otimes \cdots \otimes \rho$ is the regular repn of S_n.

- $\rho^* : R_n \longrightarrow R_{n-1}$:

Let σ be an irr. repn of S_n.

π be an irr. repn of S_{n-1}.

$$
\begin{aligned}
\langle \rho^*(\sigma), \pi \rangle &= \langle \sigma, \rho \cdot \pi \rangle \\
&= \langle \sigma, \mathrm{Ind}_{S_1 \times S_{n-1}}^{S_n} \rho \otimes \pi \rangle \\
&= \langle \sigma, \mathrm{Ind}_{S_{n-1}}^{S_n} \pi \rangle \\
&= \langle \mathrm{Res}_{S_{n-1}}^{S_n} \sigma, \pi \rangle
\end{aligned}
$$

$$\Rightarrow \quad \rho^*(\sigma) = \mathrm{Res}_{S_{n-1}}^{S_n} \sigma .$$

- $\rho^2 = \chi_2 + y_2$, χ_2 = trivial repn , y_2 = sgn repn

- $\chi_0 = 1$: trivial repn of S_0.

  $\chi_1 = \rho$ trivial repn of S_1.

  χ_2 : trivial repn of S_2,

ρ : trivial repn of S_1

χ_{n-1} : trivial repn of S_{n-1}

$$\mathrm{Ind}_{S_1 \times S_{n-1}}^{S_n} \rho \otimes \pi = \mathrm{Ind}_{S_{n-1}}^{S_n} \pi = \text{permutation repn of } S_n$$

$\ulcorner$

$$\begin{aligned} \mathbb{C} &\hookrightarrow \mathrm{Res}^{S_n}_{S_{n-1}} W \\ e_1 &\longmapsto e_1 + \cdots + e_{n-1} \end{aligned}$$

$$\Big\updownarrow$$

$$\mathrm{Ind}_{S_{n-1}}^{S_n} \mathbb{C} \longrightarrow W$$

$\lrcorner$

So $\mathrm{Ind}_{S_{n-1}}^{S_n} \pi$ = trivial repn $\oplus$ Standard repn

$\rho^*(\chi_k) = \chi_{k-1} \implies \chi_k$ = trivial repn.

- $y_0 = 1$

  $y_1 = \rho$

  $y_2 =$ sign repn of S_2

  $y_{n-1} =$ sign repn of S_{n-1}

  $\rho \cdot y_{n-1} = \mathrm{Ind}_{S_{n-1}}^{S_n} \cdot y_{n-1} =$ sign repn $\otimes$ permutation repn

  $=$ sign repn $+$ sign $\otimes$ standard

  $\rho^* (y_n) = y_{n-1} \Rightarrow y_n =$ sign repn.

- $t : R \longrightarrow R$

  $x_n \longmapsto y_n$

  $\Downarrow$

  $t : R(S) \longrightarrow R(S)$

  $\pi \longmapsto \text{sign} \otimes \pi$ (inner tensor product)

- $\sum_{k=0}^{n} (-1)^k \mathrm{Ind}_{S_k \times S_{n-k}}^{S_n} 1_{S_k} \otimes 1_{S_{n-k}} = 0$, 1_G: trivial repn of G.