

# \* The ring of symmetric functions

- ①  $R$  a ring  
 $B_R = R[\xi_1, \xi_2, \dots]$   
 $A_R \subseteq B_R$  consisting of power series that are  
 1) symmetric, 2) bounded.

$$A := A_{\mathbb{Z}}$$

$$\lambda \leftrightarrow u_\lambda$$

$$R \mapsto A_R$$

comm  
Rings

$\rightarrow$

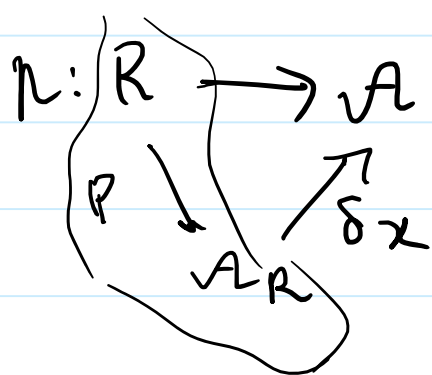
comm  
graded  
rings.

$$② A_R^N$$

$$\parallel A_R \rightarrow A_R^N$$

$$\varphi_N : A_R^{N'} \rightarrow A_R^N, N' \geq N$$

$$A_R^N \cap A_R^{N'}$$



$$p^{(N)} : R \rightarrow R[\xi_1, \dots, \xi_N]$$

$$R \xrightarrow{p^{(N)}} \underbrace{R \otimes \dots \otimes R}_{N \text{ times}} \xrightarrow{p_2} R[\xi_1] \otimes \dots \otimes R[\xi_N]$$

$$\downarrow p_3$$

$$R[\xi_1, \dots, \xi_N]$$

$p_1$ : iterating  $m^*$ .

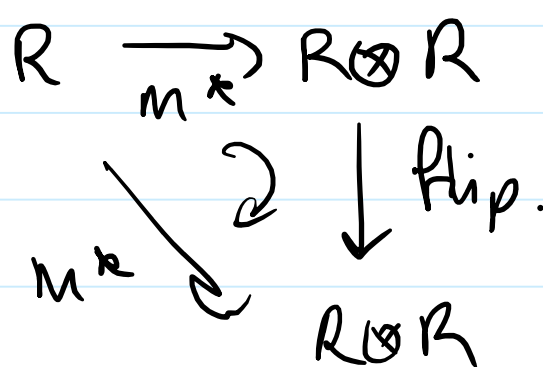
$p_2$ : tensoring the map  $R \rightarrow R[\xi_1, \dots, \xi_N]$   
 $\sigma \mapsto \sigma \xi_N$   
 $\in R_N$

$p_3$ :

$p_1, p_2, p_3$  and hence  $p^{(N)}$  are morphisms of graded rings.

claim:  $p^{(N)} = p_3 \circ p_2 \circ (p_1 : R \rightarrow R[\xi_1, \dots, \xi_N])$   
 has its image in  $A_R^N \subseteq R[\xi_1, \dots, \xi_N]$ .

pf: • base case ( $N=2$ )



• Suppose  $\text{Im } p_1^{(N)}$  is symmetric for some  $N \geq 1$ .

$$\boxed{R \otimes R \cdots \otimes R \otimes R}$$

$$R \xrightarrow{p^{(N)}} R \otimes \cdots \otimes R$$

$$\begin{array}{c} p^{(N)} \searrow \downarrow \sigma \\ R \otimes \cdots \otimes R \end{array}$$

$$\begin{array}{c} p^{(N+1)} \searrow \\ R \xrightarrow{m^*} R \otimes R \xrightarrow{p^{(N)} \otimes \text{id}} \boxed{R \otimes \cdots \otimes R \otimes R} \\ \downarrow \sigma \otimes \text{id} \\ R \otimes \cdots \otimes R \otimes R \end{array}$$

assoc.  $\Rightarrow p^{(N)}$  has image in  $\mathcal{U}_R^N$ .  $\square$

$$p^{(N)} : R \longrightarrow \mathcal{U}_R^N$$

$$p : R \longrightarrow \mathcal{A}_R$$

$$n : \underline{R} \xrightarrow{p} \mathcal{A}_R \xrightarrow{\delta_n} \underline{\mathcal{A}}$$

$$\begin{array}{c} \delta_n : R \longrightarrow \mathbb{Z} \\ \sigma \longmapsto \chi_n^*(\sigma) \\ \in R_n \end{array}$$

### EXAMPLES

$$\textcircled{1} p \in R, \quad p_1, p_2, p_3, p^{(N)} \quad \delta_n$$

$$R \longrightarrow R \otimes R$$

$$\boxed{N=2} : p_1(p) = p \otimes 1 + 1 \otimes p$$

$$\begin{array}{c} \downarrow p_2 \\ p \zeta_1 \otimes 1 + 1 \otimes p \zeta_2 \\ \downarrow p_3 \end{array}$$

$$p \in R_1$$

$$p \zeta_1 + p \zeta_2$$

$$\delta_n(p) = 1$$

$$\downarrow \delta_n$$

$$\zeta_1 + \zeta_2$$

$$R \rightarrow R \otimes R \rightarrow R \otimes R \otimes R$$

$$[N=3] : P_1(p) =$$

$$p \mapsto (p \otimes 1 + 1 \otimes p) \mapsto m^*(p) \otimes 1 + m^*(1) \otimes p$$

$$(p \otimes 1 + 1 \otimes p) \otimes 1 + 1 \otimes 1 \otimes p$$

$$p \otimes 1 \otimes 1 + 1 \otimes p \otimes 1 + 1 \otimes 1 \otimes p.$$

$$\downarrow P_2$$

$$p \zeta_1 \otimes 1 \otimes 1 + 1 \otimes p \zeta_2 \otimes 1 + 1 \otimes 1 \otimes p \zeta_3$$

$$\downarrow P_3$$

$$p \zeta_1 + p \zeta_2 + p \zeta_3$$

$$\neq [N=2]$$

$$\downarrow \delta x$$

$$\zeta_1 + \zeta_2 + \zeta_3.$$

$$R \rightarrow A$$

$$p \mapsto \zeta_1 + \zeta_2 + \zeta_3 + \dots$$

$$(2) \chi_n. \quad m^*(\chi_n) = \sum_{i=0}^n \chi_i \otimes \chi_{n-i}$$

$$P_1^{(N)}(\chi_n) = \sum_{l_1 + \dots + l_N = n} \chi_{l_1} \otimes \dots \otimes \chi_{l_N}$$

$$\text{So } P^{(N)}(\chi_n) = \sum_{l_1 + \dots + l_N = n} \underbrace{\chi_{l_1} \chi_{l_2} \dots \chi_{l_N}}_{\substack{\zeta_1^{l_1} \zeta_2^{l_2} \dots \zeta_N^{l_N}}} \zeta_1^{l_1} \zeta_2^{l_2} \dots \zeta_N^{l_N}$$

$$P(\chi_n) = \sum_{|\alpha| = n} \chi_\alpha \zeta^\alpha$$

$$\delta x(y_n) = \begin{cases} 0 & k > 2 \\ 1 & k = 1, 0 \end{cases}$$

$$\delta x$$

$$\delta x(\chi_k) = 1$$

$$R \rightarrow A$$

$$\chi_n \mapsto \sum_{|\alpha| = n} \zeta^\alpha = h_n$$

The sum of all monomials of degree  $n$ .

$$\textcircled{3} \ x_{\lambda}^{\perp} \quad \lambda \in \mathcal{P}, \quad x_{\lambda} := x_{l_1} x_{l_2} \dots x_{l_r}.$$

"  $(l_1, \dots, l_r)$

These  $x_{\lambda}$ s form a basis.

$x_{\lambda}^{\perp}$  is defined as dual basis to  $x_{\lambda}$ s.

want  $P(x_{\lambda}^{\perp})$ ,  $v \in R_n$ ,

$$P(v) = \sum_{\lambda \in \mathcal{P}_n} x_{\lambda}^*(v) \cdot u_{\lambda} = \sum_{\lambda \in \mathcal{P}_n} \langle x_{\lambda}, v \rangle u_{\lambda}.$$

$$P(x_{\lambda}^{\perp}) = u_{\lambda}$$

$P: R \longrightarrow A$

$$\lambda \in \mathcal{P}_n, \quad x_{\lambda} \in R_n, \quad v \in R_n \quad \left| \quad \begin{array}{l} \langle ab, c \rangle = \\ \langle a \otimes b, m^*(c) \rangle \end{array} \right.$$

$\langle x_{\lambda}, v \rangle = x_{\lambda}^*(v)$

And,

$$\langle x_{\lambda}, v \rangle = \langle x_{l_1} \dots x_{l_n}, v \rangle$$

$$= \langle x_{l_1} \otimes \dots \otimes x_{l_n}, p^{(n)}(v) \rangle$$

$$= \langle x_{l_1} \otimes \dots \otimes x_{l_n}, v_{(1)} \otimes \dots \otimes v_{(n)} \rangle, \quad v_{(i)} \in R_{l_i}$$

$$= \langle x_{l_1}, v_{(1)} \rangle \langle x_{l_2}, v_{(2)} \rangle \dots \langle x_{l_n}, v_{(n)} \rangle$$

$$= x_{l_1}^*(v_{(1)}) \dots x_{l_n}^*(v_{(n)})$$

$$= \delta x(v_{(1)}) \dots \delta x(v_{(n)}).$$

$$\lambda = (l_1, \dots, l_n) \quad u_{\lambda}.$$

$$P(v) = \sum_{\lambda \in \mathcal{P}_n} \langle x_{\lambda}, v \rangle u_{\lambda} = \sum_{\substack{l_1, l_2, \dots, l_n \\ v_{(i)} \in R_{l_i}}} x_{l_1}^*(v_{(1)}) x_{l_2}^*(v_{(2)}) \dots x_{l_n}^*(v_{(n)})$$

Prop:  $R \longrightarrow A$  is an iso.

$$x_{\lambda}^{\perp} \longmapsto u_{\lambda}.$$

Schur Functions

$$R \xrightarrow{\quad} \sqrt{\lambda}$$

 $\langle, \rangle$ 

Section 4:  $\lambda \in P$ ,  $\exists!$   $\{\lambda\} \in R_{1\lambda}$  irreducible element s.t.

$$\kappa_{\lambda}^*(\{\lambda\}) \neq 0, \quad y_{\lambda^+}^*(\{\lambda\}) \neq 0$$

$$\text{and } \kappa_{\lambda}^*(\{\lambda\}) = y_{\lambda^+}^*(\{\lambda\}) = 1.$$

$$\lambda \mapsto \{\lambda\} \text{ is bij } P \longrightarrow \sqrt{\lambda}$$

The image of  $\{\lambda\}$  are called Schur functions.

---