* The ring of symmetric functions

① R a ring

$B_R = R[[\xi_1, \xi_2, \dots]]$

$A_R \subseteq B_R$ consisting of power series that are
1) symmetric, 2) bounded.

$A := A_{\mathbb{Z}}$

$\lambda \leftrightarrow u_\lambda$

$R \longmapsto A_R$ comm Rings $\longrightarrow$ comm graded rings.

② A_R^N

$\| \; A_R \longrightarrow A_R^N$

$\varphi_N : A_R^{N'} \longrightarrow A_R^N \;, N' \geq N$

A_R^N
$\cap$

$\rho : R \longrightarrow A$, $p : R \to A_R$, $\delta_x : A_R \to A$

$p^{(N)} : R \longrightarrow \underline{R[\xi_1, \dots, \xi_N]}$

$R \xrightarrow{p_1^{(N)}} \underline{R \otimes \dots \otimes R} \xrightarrow{p_2} R[\xi_1] \otimes \dots \otimes R[\xi_N]$

$\downarrow p_3$

$R[\xi_1, \dots, \xi_N]$.

p_1 : iterating m^*.

p_2 : tensoring the map $R \to R[\xi_1]$
$\sigma \longmapsto \sigma \xi^n$
$\in R_n$

p_3 :

p_1, p_2, p_3 and hence $p^{(N)}$ are morphisms of graded rings.

claim: $p^{(N)} = p_3 \circ p_2 \circ p_1 : R \to R[\xi_1, \dots, \xi_N]$
has its image in $A_R^N \subseteq R[\xi_1, \dots, \xi_N]$.

Pf: • base case $(N = 2)$

$$\begin{tikzcd} R \arrow[r, "m^*"] \arrow[dr, "m^*"'] & R \otimes R \arrow[d, "\text{flip}"] \\ & R \otimes R \end{tikzcd}$$

• Suppose $\operatorname{Im} p_1^{(N)}$ is symmetric for some $N \geq 1$.

$$\boxed{R \otimes R \cdots \otimes R \otimes R}$$

$$\begin{tikzcd}
R \arrow[r, "p_1^{(N)}"] \arrow[dr, "p_1^{(N)}"'] & R \otimes \cdots \otimes R \arrow[d, "\sigma"] \\
 & R \otimes \cdots \otimes R
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, "m^*"] \arrow[rr, bend left, "p_1^{(N+1)}"] \arrow[rrd, bend right, "p_1^{(N+1)}"'] & R \otimes R \arrow[r, "p_1^{(N)} \otimes \mathrm{id}"] \arrow[dr, "p_1^{(N)} \otimes \mathrm{id}"'] & R \otimes \cdots \otimes R \otimes R \arrow[d, "\sigma \otimes \mathrm{id}"] \\
 & & R \otimes \cdots \otimes R \otimes R
\end{tikzcd}$$

coassoc $\Rightarrow$ $p^{(N)}$ has image in $\mathcal{A}_R^N$. $\square$

$p^{(N)} : R \to \mathcal{A}_R^N$.

$p : R \to \mathcal{A}_R$

$\chi : \underline{R} \xrightarrow{p} \mathcal{A}_R \xrightarrow{\delta_x} \underline{\mathcal{A}}$

$\delta_x : R \to \mathbb{Z}$

$v \mapsto x_n^*(v)$.

$\in R_n$

$\boxed{\text{EXAMPLES}}$

$p_1, p_2, p_3, p^{(N)}$ δ_x

(1) $\rho \in R$.

$R \to R \otimes R$

$\boxed{N=2}$: $p_1(\rho) = \rho \otimes 1 + 1 \otimes \rho$

$\downarrow p_2$ $\rho \in R_1$

$\rho \xi_1^1 \otimes 1 + 1 \otimes \rho \xi_2^1$

$\downarrow p_3$

$\rho \xi_1 + \rho \xi_2$ $\delta_x(\rho) = 1$

$\downarrow \delta_x$

$\xi_1 + \xi_2$.

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend right] & R\otimes R \arrow[r] & R\otimes R\otimes R \end{tikzcd}$$

$\boxed{N=3}$: $P_1(\rho) =$

$$\rho \mapsto (\rho\otimes 1 + 1\otimes \rho) \mapsto m^*(\rho)\otimes 1 + m^*(1)\otimes \rho$$

$$= (\rho\otimes 1 + 1\otimes\rho)\otimes 1 + 1\otimes 1\otimes \rho$$

$$= \rho\otimes 1\otimes 1 + 1\otimes\rho\otimes 1 + 1\otimes 1\otimes \rho.$$

$$\downarrow P_2$$

$$\rho\xi_1\otimes 1\otimes 1 + 1\otimes\rho\xi_2\otimes 1 + 1\otimes 1\otimes\rho\xi_3$$

$$\downarrow P_3$$

$$\rho\xi_1 + \rho\xi_2 + \rho\xi_3 \qquad \boxed{N=2}$$

$$\downarrow \delta_x$$

$$\xi_1 + \xi_2 + \xi_3.$$

$$R \longrightarrow A$$

$$\rho \mapsto \xi_1 + \xi_2 + \xi_3 + \cdots$$

(2) x_n. $\boxed{m^*(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}}$

$$P_1^{(N)}(x_n) = \sum_{l_1+\cdots+l_N=n} x_{l_1}\otimes \cdots \otimes x_{l_N}$$

So $\rho^{(N)}(x_n) = \sum_{l_1+\cdots+l_N=n} x_{l_1}x_{l_2}\cdots x_{l_N}\,\xi_1^{l_1}\xi_2^{l_2}\cdots\xi_N^{l_N}$

$$P(x_n) = \sum_{|\alpha|=n} x_\alpha \xi^\alpha \qquad \delta_x(y_k) = \begin{cases} 0 & k \geq 2 \\ 1 & k=1,0 \end{cases}$$

δ_x

$\underline{\delta_x}(x_k) = 1$

$$R \longrightarrow A$$

$$x_n \mapsto \sum_{|\alpha|=n}\xi^\alpha = h_n$$

The sum of all monomials of degree n.

(3) $x_\lambda^\perp$. $\lambda \in \mathcal{P}$, $x_\lambda := x_{l_1} x_{l_2} \cdots x_{l_r}$.
$\lambda = (l_1, \dots, l_r)$

These x_λs form a basis.

$x_\lambda^\perp$ is defined as dual basis to x_λs.

Want $P(x_\lambda^\perp)$; $v \in R_n$,

$$P(v) = \sum_{\lambda \in \mathcal{P}_n} x_\lambda^*(v) \cdot u_\lambda = \sum_{\lambda \in \mathcal{P}_n} \langle x_\lambda, v \rangle u_\lambda .$$

$$P(x_\lambda^\perp) = u_\lambda$$

$$P : R \longrightarrow \mathcal{A}$$

$\lambda \in \mathcal{P}_n$, $x_\lambda \in R_n$, $v \in R_n$ $\langle x_\lambda, v \rangle = x_\lambda^*(v)$.

$\langle ab, c \rangle = \langle a \otimes b, m^*(c) \rangle$

And,
$$\langle x_\lambda, v \rangle = \langle x_{l_1} \cdots x_{l_N}, v \rangle$$
$$= \langle x_{l_1} \otimes \cdots \otimes x_{l_N}, \rho^{(N)}(v) \rangle$$
$$= \langle x_{l_1} \otimes \cdots \otimes x_{l_N}, v_{(1)} \otimes \cdots \otimes v_{(N)} \rangle, \quad v_{(i)} \in R_{l_i}$$
$$= \langle x_{l_1}, v_{(1)} \rangle \langle x_{l_2}, v_{(2)} \rangle \cdots \langle x_{l_N}, v_{(N)} \rangle$$
$$= x_{l_1}^*(v_{(1)}) \cdots x_{l_N}^*(v_N)$$
$$= \delta_x(v_{(1)}) \cdots \delta_x(v_{(N)}).$$

P.

$\lambda = (l_1, \dots, l_N)$. u_λ.

$$P(v) = \sum_{\lambda \in \mathcal{P}_n} \langle x_\lambda, v \rangle u_\lambda$$

$\xi_1^{l_1} \xi_2^{l_2} \cdots \xi_N^{l_N}$

$v_{(i)} \in R_{l_i}$

<u>Prop</u>: $R \to \mathcal{A}$ is an iso.
$$x_\lambda^\perp \longmapsto u_\lambda .$$

Section 5 Notes (P8)

Wednesday, 17 March 2021 1:08 PM

Schur Functions

$R \longrightarrow \Lambda$ $\qquad\qquad \langle , \rangle$

Section 4: $\lambda \in \mathcal{P}$, $\exists!$ $\{\lambda\} \in R_{|\lambda|}$ irreducible element s.t.

$$x^*_\lambda(\{\lambda\}) \neq 0, \quad y^*_{\lambda^t}(\{\lambda\}) \neq 0$$

and $x^*_\lambda(\{\lambda\}) = y^*_{\lambda^t}(\{\lambda\}) = 1.$

$\lambda \mapsto \{\lambda\}$ is bij $\mathcal{P} \longrightarrow \Omega$

The image of $\{\lambda\}$ are called Schur functions.