Thm: Let R be a PSH-algebra with unique irreducible primitive element ρ of degree 1.

1). There exist unique irred. elements

(x_n), (y_n) in R_n, $n \in \mathbb{Z}_{\geq 0}$

such that

- $\sum_{k=0}^{n} (-1)^k x_k \cdot y_{n-k} = 0 \quad (n \geq 1)$

- $R = \mathbb{Z}[x_1, x_2, \cdots]$

- $R = \mathbb{Z}[y_1, y_2, \cdots]$.

2). R has a unique non-trivial automorphism t such that

$$t(x_n) = y_n \qquad n \geq 1.$$
$$t(y_n) = x_n$$

3). Any PSH-algebra R' with unique irr. primitive element p' of degree 1 is isomorphic to R as PSH-algebra. There are exactly two PSH-algebra isomorphism from R to R'.

lemma1: p^2 is a sum of two distinct irr elements. : $p^2 = x_2 + y_2$.

Pf : $\langle p^2, p^2 \rangle = 2$

lemma 2 : For all $n \in \mathbb{Z}_{\geq 0}$, $\exists!$ irr elements $x_n, y_n \in R_n$ such that

$$x_2^*(y_n) = 0, \quad y_2^*(x_n) = 0.$$

证: Prove

the desired x_n, y_n exist, are unique

and satisfy

$$\rho^*(x_n) = x_{n-1}, \quad \rho^*(y_n) = y_{n-1} \quad (n \geq 1).$$

Base case:

$n=0$: $\quad x_2^*(1) = y_2^*(1) = 0.$

$n=1$: $\quad x_2^*(\rho) = y_2^*(\rho) = 0$

$\rho^*(\rho) = 1$

Induction step: let $k \in \mathbb{Z}_{\geq 2}$

assume that x_n, y_n are constructed

for $n < k$.

<u>Claim 1</u>: If x_k satisfies the desired property

then $\langle x_k, \rho \cdot x_{k-1} \rangle > 0$.

Claim 2: $\rho \cdot X_{k-1} \in R_k$ is a sum of two distinct irred elements.

Claim 3: $y_2^* (\rho \cdot X_{k-1}) = X_{k-2}$.

$\Rightarrow$ y_2^* takes one irr. constituent $\rho \cdot X_{k-1}$ to 0 and the other to X_{k-2}.

X_k , y_k

Lemma : If $0 \le k \le n$, then

$$X_k^*(X_n) = X_{n-k}, \qquad Y_k^*(Y_n) = Y_{n-k}.$$

If $w \in \Omega$ is distinct from

$$x_0, \dots, x_n$$

then $w^*(x_n) = 0$.

If $w \in \Omega$ is distinct from

$$y_0, \dots, y_n$$

then $w^*(y_n) = 0$.

證: $\rho^*(x_n) = x_{n-1}$.

lemma 4 : For $n \geq 1$,

$$m^*(x_n) = \sum_{k=0}^{n} x_k \otimes x_{n-k}$$

$$m^*(y_n) = \sum_{k=0}^{n} y_k \otimes y_{n-k}$$

- Set $x_k = y_k = 0 \quad \forall \; k < 0$.

We have

$$(*) \quad \begin{cases} x_n^*(ab) = \sum_{k+l=n} x_k^*(a) \cdot x_l^*(b) \\ \qquad\qquad\qquad\qquad\qquad\qquad a, b \in R. \\ y_n^*(ab) = \sum_{k+l=n} y_k^*(a) \cdot y_l^*(b) \end{cases}$$

Define X^*, Y^* on R by

$$X^* = \sum_k x_k^* \qquad Y^* = \sum_k y_k^*$$

$(*) \Rightarrow X^*, Y^*$ are ring homomorphisms

· Define linear forms δ_x, δ_y from

$R \longrightarrow \mathbb{Z}$ by

$\delta_x(a) = x_n^*(a)$, $\delta_y(a) = y_n^*(a)$

$a \in R_n$.

<u>Note</u>: 1). δ_x, δ_y are positive.

2). δ_x, δ_y are ring homomorphisms.

3). $\delta_x(p) = \delta_y(p) = 1$.

<u>Prop</u>: Let $\delta : R \to \mathbb{Z}$ be positive, multiplicative and $\underbrace{\text{normalized}}_{\delta(p)=1}$ form.

Then either $\delta = \delta_x$ or $\delta = \delta_y$.

- Partion :

$$\mathcal{P} := \{(l_1, \cdots, l_r) \mid r \in \mathbb{N},\ l_i \in \mathbb{Z}_{\geq 0}\} / \sim$$

$\sim$: two tuples are identified if they differ by an order or the number of zeros.

elements in $\mathcal{P}$ are called partitions.

- Let $\lambda \in \mathcal{P}$.

For $k \geq 1$, let $\underset{r_k}{\overset{\shortparallel}{r_k(\lambda)}}$ be the number of parts of λ which are equal to k.

$$\lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \cdots)$$

Put $r(\lambda) = \sum_{k \geq 1} r_k(\lambda)$.

For each $\lambda = (l_1, \dots, l_r)$, let

$$|\lambda| = l_1 + \dots + l_r$$

For each $n \geq 0$, $\mathcal{P}_n = \{ \lambda \in \mathcal{P} \mid |\lambda| = n \}$

Each $\lambda \in \mathcal{P}$ can be written as

$c.f.(\lambda) = (l_1, \dots, l_r)$

where $l_1 \geq l_2 \geq \dots \geq l_r$. $l_s = 0$

for $s > r$.

- box diagram :

A box diagram is a finite subset of $\mathbb{N} \times \mathbb{N}$ containing with each point (i, j)

all points (i', j') such that $i' \leq i$, $j' \leq j$.

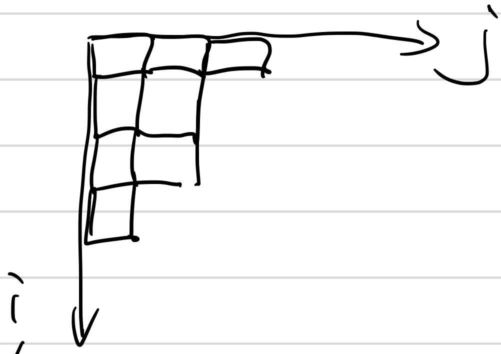
- Assign to a partition $\lambda \in \mathcal{P}$ with

$$c.f.(\lambda) = (l_1, l_2, \cdots)$$

the box diagram

$$\{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq l_i \}$$

$$(3, 2, 2, 1) \longmapsto$$



$$(0) \longmapsto \emptyset$$

This gives a bijection between $\mathcal{P}$ and set of all box diagrams.