Note :

$$e : \mathbb{Z} \to R(S)$$
$$1 \mapsto 1$$

$$e^* : R(S) \longrightarrow \mathbb{Z}$$
$$R(S) = \bigoplus_{n \geq 0} R(S_n)$$
$$\bigoplus_{n > 0} R(S_n) \longmapsto 0$$
$$R(S_0) \xrightarrow{id} \mathbb{Z}$$

e, e^* are mutually inverse isomorphism.

$$e^*|_{R(S_0)} \circ e = id, \quad e \circ e^*|_{R(S_0)} = id.$$

$\Rightarrow$ $R(S)$ is a connected Hopf algebra

Defn : A Hopf algebra R is cocommutative

if

$$\begin{tikzcd} & R \otimes R \\ R \arrow[ur, "m^*"] \arrow[r, "m^*"'] \arrow[ur, phantom, "\circlearrowright"] & R \otimes R \arrow[u, "\sigma"'] \end{tikzcd} \qquad \begin{array}{c} \sigma(x \otimes y) \\ \shortparallel \\ y \otimes x. \end{array}$$

$\Rightarrow$ $R(S)$ is cocommutative.

Prop : $R(S)$ is free $\mathbb{Z}$-module with basis $\Omega = \{ \text{irr repns of } S_n, n \geq 0 \}$.

$$\langle \omega, \tau \rangle = \delta_{\omega,\tau} = \begin{cases} 1 & \text{if } \omega = \tau \\ 0 & \text{otherwise.} \end{cases}$$

$$R(S)_+ = \left\{ \sum_{\omega \in \Omega} m_\omega \cdot \omega \;\middle|\; \begin{array}{l} m_\omega = 0 \text{ for all but finitely many } \omega \\ \text{and if } m_\omega \neq 0,\ m_\omega > 0 \end{array} \right\}$$

: positive elements of $R(S)$.

Defn : A Hopf algebra R is called positive if 1) each R_n is a free $\mathbb{Z}$-module with a basis $\Omega(R_n)$

2). m^*, m, e, e^* are positive homo.

map positive elements to positive elements

「 Frobenius Reciprocity :

G : finite group , H : subgroup of G.

$\langle \pi, \sigma \rangle_G := \dim \mathrm{Hom}_G(\pi, \sigma)$.

Let π be repn of G and ρ

repn of H.

$\langle \rho, \mathrm{Res}_H^G \pi \rangle_H = \langle \mathrm{Ind}_H^G \rho, \pi \rangle_G$. 」

(m, m^*) is an adjoint pair.

(e, e^*) is an adjoint pair.

i.e.,

Let $\rho \in R(S_n)$

Let $\pi \in R(S_k)$, $\sigma \in R(S_l)$, $k+l=n$.

$\langle \pi \otimes \sigma, m^*(\rho) \rangle = \langle m(\pi \otimes \sigma), \rho \rangle$

$\Rightarrow$ $R(S)$ is a self-adjoint Hopf algebra.

Defn : A PSH-algebra is a connected positive, self-adjoint Hopf algebra.

$\Rightarrow R(S)$ is a PSH-algebra.

- R is connected $\Rightarrow$ $R_0 \overset{\wedge}{=} \mathbb{Z}$.

Assume $R_0 = \mathbb{Z}$ and 1 is an irreducible element of R.

- Let $I = \bigoplus_{n>0} R_n$.
- Prop : For $x \in I$,

$$m^*(x) = 1 \otimes x + x \otimes 1 + m_+^*(x)$$

where $m_+^*(x) \in I \otimes I$.

Defn : An element x in I is called primitive if $m_+^*(x) = 0$.

$$m^*(x) = 1 \otimes x + x \otimes 1.$$

P : primitive elements in R.

<u>Thm</u> : Any PSH algebra R decomposes into the tensor product of PSH algebra with only one irreducible primitive element.

How the decomposition is done :

$C = \Omega \cap P$.

For $\rho \in \Omega \cap P$

$$\Omega(\rho) = \{ \omega \in \Omega \mid \langle \omega, \rho^n \rangle \text{ for some } n \geq 0 \}$$

$$R(\rho) = \bigoplus_{\omega \in \Omega(\rho)} \mathbb{Z} \cdot \omega .$$

$$R = \bigotimes_{\rho \in C} R(\rho) .$$

Let $\sigma \in R(S)$ be an irr primitive element.

Then $m^*(\sigma) = 1 \otimes \sigma + \sigma \otimes 1$.

If $\sigma \in R(S_n)$ with $n > 1$,

$$m^*(\sigma) = \sum_{k+l=n} \mathrm{Res}^{S_n}_{S_k \times S_l}(\sigma)$$

then σ is not primitive

So $\sigma \in R(S_1)$.

Since $R(S_1)$ is of rank 1,

$R(S)$ has only one irreducible primitive element.

- R : PSH - algebra.

For any $x \in R$, denote by

$$x^* : R \longrightarrow R$$

the operator adjoint to that of multiplication by x,

$$x^* : R \longrightarrow R$$
$$y \longmapsto x^*(y)$$

$$\langle x^*(y), z \rangle = \langle y, x \cdot z \rangle$$

for all $y, z \in R$.

Eg: $R(S)$.

If $x = 1$, then $x^* = id_R$.

For irred.

Prop : a). $x \in R_k$ Then

$$x^*(R_n) \subset R_{n-k} \quad \text{for } n \geq k$$

$$x^*(R_n) = 0 \quad \text{for } n < k.$$

b). $x^* : R \longrightarrow R$ equals to the composition

$$R \xrightarrow{m^*} R \otimes R \xrightarrow{id \otimes \langle x, - \rangle} R \otimes \mathbb{Z} \xrightarrow{\sim} R$$

c). For any $x, y \in R$

$$(xy)^* = y^* \circ x^*$$

d). $x \in R^+ \Rightarrow x^*$ is positive.

e). $x, y, z \in R$ and

$$m^*(x) = \sum_i a_i \otimes b_i$$

Then $$x^*(yz) = \sum_i a_i^*(y) \otimes b_i^*(z).$$

f). If $p \in R$ is primitive, then

$$p^* : R \longrightarrow R$$

is a derivation of R, i.e.

$$p^*(yz) = p^*(y) \cdot z + y \cdot p^*(z).$$

g). If $\rho \in R_n$ is primitive, $0 < k < n$, $x \in R_k$, then $x^*(\rho) = 0$.

R: PSH-algebra with the unique irr. primitive element ρ.

From the decomp thm, any irr element $w \in R$ is an irr constituent of ρ^n

i.e. $\langle w, \rho^n \rangle > 0$ for some $n \geq 0$

In particular, if $R_m \neq 0$, then

$$\deg \rho \mid m.$$

So we can change grading on R by dividing all degrees by $\deg \rho$

We assume. $R_1 = \mathbb{Z} \cdot \rho$.

<u>Thm</u> : If R is a PSH-algebra with the unique irreducible primitive element ρ which is of degree 1, then there exist

$$x_1, x_2, \cdots \in R .$$

such that as rings,

$$R = \mathbb{Z}[x_1, x_2, \cdots] .$$

There exist $y_1, y_2, \cdots \in R$ such that

$$\sum_{k=0}^{n} (-1)^k x_k \cdot y_{n-k} = 0 \qquad (n \geq 1)$$

Moreover, $R = \mathbb{Z}[y_1, y_2, \cdots]$