Let A be ring.

<u>Defn</u>: Let $\mathcal{C}$ be a category of left A-module. The Grothendieck group K of $\mathcal{C}$ is an abelian defined as the following

generators: $[E]$, E: left A-module.

relation: $[E] = [V] + [U]$ if there exists a short exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow U \longrightarrow 0.$$

Eg: A: field.

From now on, let $A = \mathbb{C}[S_n]$.

Let S be the set of simple A-module.

Then l is a free abelian group with basis B

$R(S_n)$: Grothendieck group of category of f-dim repns of S_n.

Let $R(S) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R(S_n)$.

⌜ Let π be a repn of S_n

Then $\pi = \sum_{\sigma \in B} \langle \pi, \sigma \rangle_{S_n} \cdot \sigma$

$$\langle \pi, \sigma \rangle_{S_n} = \dim \mathrm{Hom}_{S_n}(\pi, \sigma)$$

$$\langle \rho, \sigma \rangle_{S_n} = \begin{cases} 1 & \rho \cong \sigma \\ 0 & \text{otherwise.} \end{cases}$$

⌟

1)
Now, we define multiplication

$$m : R(S) \otimes R(S) \longrightarrow R(S).$$

「 Induced repn :

①

Let G be group and H its subgroup.

Let (ρ, W) be a repn of H.

W : $\mathbb{C}[H]$-module.

$\mathrm{Ind}_H^G W := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$.

②. Let R = set of representatives of left coset of H.

Let $\mathrm{Ind}_H^G W := \bigoplus_{s \in R} s \cdot W$

where each $s \cdot W$ is a vector space consisting of vectors of the form $s \cdot w$

Let $g \in G$, $\sum_{s_i \in R} s_i w_i$

Then $g \cdot \sum_{s_i \in R} s_i w_i := \sum_{s_i \in R} s_{j(i)} \cdot \rho(h) \cdot w_i$

where $g \cdot s_i = s_{j(i)} \cdot h$ 」

Let $n \in \mathbb{Z}_{\geq 0}$ and $k, l \in \mathbb{Z}_{\geq 0}$ such that

$$k + l = n.$$

Then $S_k \times S_l \subset S_n$

Let σ be a irr repn S_k and

π be a irr repn S_l.

Then $\sigma \otimes \pi$ is an irr. $S_k \times S_l$.

$\mathrm{Ind}_{S_k \times S_l}^{S_n} \sigma \otimes \pi$ is a repn of S_n.

Define

$$m : R(S) \otimes R(S) \longrightarrow R(S)$$

$$\sigma \otimes \pi \longmapsto \mathrm{Ind}_{S_k \times S_l}^{S_n} \sigma \otimes \pi.$$

2). Unit.

$$e : \mathbb{Z} \longrightarrow R(S)$$

$$1 \longmapsto 1$$

3). Comultiplication

$$m^*: R(S) \longrightarrow R(S) \otimes R(S).$$

Let σ be an irr repn of S_n.

Let $k, l \in \mathbb{Z}_{\geq 0}$ s.t $k + l = n$.

We have $S_k \times S_l \subset S_n$.

$\mathrm{Res}^{S_n}_{S_k \times S_l} \sigma$ is a repn of $S_k \times S_l$.

Define $$m^*(\sigma) := \sum_{k+l=n} \mathrm{Res}^{S_n}_{S_k \times S_l} \sigma.$$

4). Counit

$$\begin{aligned} e^*: R(S) &\longrightarrow \mathbb{Z} \\ \| \quad & \\ \bigoplus_{n \geq 0} R(S_n) & \\ \bigoplus_{n \geq 1} R(S_n) &\longmapsto 0 \\ \mathbb{Z} = R(S_0) &\xrightarrow{id} \mathbb{Z}. \end{aligned}$$

1). m is associative.

2). $e(1) = 1$

3). m^* is coassociative :

$$\begin{tikzcd}
R(S) \arrow[r, "m^*"] \arrow[d, "m^*"'] \arrow[dr, phantom, "\circlearrowright"] & R(S) \otimes R(S) \arrow[d, "id \otimes m^*"] \\
R(S) \otimes R(S) \arrow[r, "m^* \otimes id"'] & R(S) \otimes R(S) \otimes R(S)
\end{tikzcd}$$

4). e^* satisfies :

$$\begin{tikzcd}
 & R(S) \otimes R(S) \arrow[dl, "e^* \otimes id"'] \arrow[dr, "id \otimes e^*"] & \\
\mathbb{Z} \otimes R(S) \arrow[r, phantom, "\circlearrowright"] & R(S) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "m^*"'] & R(S) \otimes \mathbb{Z} \arrow[l, phantom, "\circlearrowright"]
\end{tikzcd}$$

e.g. Let $\sigma \in R(S_1)$ be an irreducible element

$m^*(\sigma) = 1 \otimes \sigma + \sigma \otimes 1$. Then $e^* \otimes id(m^*(\sigma))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad id \otimes e^*(m^*(\sigma))$

5). m^* is a ring homomorphism:

$$\begin{tikzcd}
R(S)\otimes R(S) \arrow[r, "m"] \arrow[d, "m^*\otimes m^*"'] & R(S) \arrow[r, "m^*"] \arrow[d, phantom, "\circlearrowright"] & R(S)\otimes R(S) \\
(R(S)\otimes R(S))\otimes(R(S)\otimes R(S)) \arrow[d, "m"'] & & \\
R(S)\otimes R(S) \arrow[uurr, equal, "\mathrm{id}"'] & &
\end{tikzcd}$$

7 Mackey Theorem. (ρ, W): repn of H

G : group H, K : subgroups of G.

Let $K\backslash G/H$ denote the double cosets of G.

Let R be the set of representatives.

$$\left(G = \coprod_{s\in R} KsH \right)$$

For each $s\in R$, set $H_s = sHs^{-1}\cap K \subset K$

Consider $\quad \rho^s : H_s \longrightarrow GL(W)$

$$x \longmapsto \rho(s^{-1} x s)$$

Let $\quad W_s = (\rho^s, W)$ denote corresponding repn of H_s

Then $$\mathrm{Res}_K^G \mathrm{Ind}_H^G W = \bigoplus_{s \in R} \mathrm{Ind}_{H_s}^K W_s .$$

Thus, $R(S)$ has a Hopf algebra structure.