Hopf algebras and primitive elements

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Outline

1. Define Hopf algebras.
2. Define PSH algebras.
3. Discuss basic properties of PSH algebras

Graded rings

Definition

A ring K is called *graded* if it can be expressed as a direct sum of abelian groups

$$K = \bigoplus_{n=0}^{\infty} K_n$$

such that $K_n K_m \subset K_{n+m}$

Any ring is a graded ring with $K_0 = K$ and $K_n = 0$ for $n \geq 1$.

Graded modules

Definition

Let K be a graded ring. A K-module R is called a *graded module* if R can be expressed as a direct sum of abelian groups

$$R = \bigoplus_{n=0}^{\infty} R_n$$

such that $K_n R_m \subset R_{n+m}$. Elements of R_n are called *homogeneous of degree* n.

A morphism of graded K-modules is a K-module morphism $f : M \to N$ with $f(M_i) \subseteq N_i$

Hopf algebras

Let K be a commutative ring.

Definition

A *Hopf algebra over* K is a graded K-module $R = \bigoplus_{n\geq 0} R_n$ with graded K-module morphisms $m : R \otimes R \to R$ (multiplication), $m^* : R \to R \otimes R$ (comultiplication), $e : K \to R$ (unit) and $e^* : R \to K$ (counit) satisfying:

- m is associative and m^* is coassociative.
- e is a unit of R and e^* is a counit.
- (Hopf Axiom). The operator $m^* : R \to R \otimes R$ is a ring homomorphism.

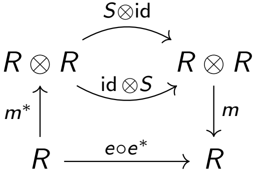
i.e. **Associativity** **Counit**

$$\begin{array}{ccc} R\otimes R\otimes R & \xrightarrow{m\otimes \mathrm{id}} & R\otimes R \\ \downarrow{\scriptstyle \mathrm{id}\otimes m} & & \downarrow{\scriptstyle m} \\ R\otimes R & \xrightarrow{m} & R \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{m^*} & R\otimes R \\ \downarrow{\scriptstyle m^*} & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \mathrm{id}\otimes e^*} \\ R\otimes R & \xrightarrow{e^*\otimes \mathrm{id}} & K\otimes R \cong R \cong R\otimes K \end{array}$$

Hopf algebras

Note: Another definition for Hopf algebra requires the existence of an "antipode" i.e. a K-linear isomorphism $S : R \to R$ such that the following diagram commutes



To remedy this disparity, we can define an antipode recursively via $S(x) = -x$ for $x \in R_1$ and $S(x) = -x - m \circ (\mathrm{id} \otimes S) \circ m_+^*(x)$ for $x \in R_n$, $n > 1$. (See Chapter 6 Proposition 3.9 of Gruson & Serganova (2018))

Example

- Let G be a group. Then the group algebra kG has a Hopf algebra structure with

$$m^*(g) = g \otimes g, \qquad e^*(g) = 1$$

- The polynomial algebra $k[X]$ has a Hopf algebra structure with comultiplication induced by

$$m^*(X) = X \otimes 1 + 1 \otimes X$$

and counit induced by

$$e^*(X) = 0$$

Properties of Hopf algebras

Definition

A Hopf algebra R is called *connected* if $e : K \to R_0$ and $e^* : R_0 \to K$ are mutually inverse isomorphisms.

Properties of Hopf algebras

Definition

A T-group is a free $\mathbb{Z}$-module with a distinguished $\mathbb{Z}$-basis $\Omega = \Omega(R)$. Elements of $\Omega(R)$ are called *basic elements* of R.

If R is a T-group then put

$$R^+ = \{\sum_{\omega \in \Omega} m_\omega \omega \mid m_\omega \geq 0\}$$

Elements of R^+ are called *positive*. A homomorphism of two T-groups is called *positive* if it takes positive elements to positive elements.

Definition

A Hopf algebra R over $\mathbb{Z}$ is called *positive* if it satisfies

1. Each R_n and hence the whole R is a T-group.
2. The morphisms m, m^*, e, and e^* are positive.

Properties of Hopf algebras

We can define a $\mathbb{Z}$-valued bilinear form $\langle , \rangle$ on R by

$$\langle w, w' \rangle = \delta_{\omega,\omega'}$$

for $\omega, \omega' \in \Omega$. $\langle , \rangle$ is symmetric, nondegenerate and positive definite. We will refer to $\langle , \rangle$ as the "inner product" induced by the T-group structure.

Definition

A positive Hopf algebra is called *self-adjoint* if m and m^* (respectively e and e^*) are adjoint to each other with respect to inner products on R, $R \otimes R$ and $\mathbb{Z}$ induced by a T-group structure.

PSH algebras

Definition

A *PSH algebra* R is a connected positive self-adjoint Hopf algebra over $\mathbb{Z}$.

Example

Let S_n be the permutation group and $R(S_n)$ be the Grothendieck group of the category of finite dimensional complex representations of S_n. Then one can define a Hopf algebra structure on $\bigoplus_{n\geq 0} R(S_n)$ and one can prove that it is a PSH algebra. For more details, see 1.1 in Zelevinsky (1981).

Properties of PSH algebras

Let $R = \bigoplus_{n \geq 0} R_n$ be a PSH algebra.
We denote the unit of R i.e. the element $e(1) \in R_0$ by 1.
Set $I = \bigoplus_{n>0} R_n$.

Proposition

For $x \in I$ we have

$$m^*(x) = x \otimes 1 + 1 \otimes x + m^*_+(x)$$

where $m^*_+(x) \in I \otimes I$

Properties of PSH algebras

Proof.

Since R is connected and e^* is a graded morphism, we have $e^*(I) = 0$. Now write

$$m^*(x) = y \otimes 1 + 1 \otimes z + m^*_+(x)$$

where $y, z \in I$ and $m^*_+(x) \in I \otimes I$.
By the counit property, we have

$$(e^* \otimes \mathrm{id})(m^*(x)) = 1 \otimes x$$

and

$$(\mathrm{id} \otimes e^*)(m^*(x)) = x \otimes 1$$

Combining, we get $x = y = z$.

□

An element $x \in I$ is called *primitive* if $m^*_+(x) = 0$. Denote by P the subgroup of primitive elements in R.

Properties of PSH algebras

For any $x \in R$ denote by $x^* : R \to R$ the operator adjoint to multiplication i.e. x^* is defined by

$$\langle x^*(y), z \rangle = \langle y, xz \rangle$$

Denote by $\langle x \mid$ the map

$$\langle x \mid : R \to \mathbb{Z}$$

given by

$$y \mapsto \langle x, y \rangle$$

Properties of PSH algebras

Proposition

a) *If* $x \in R_k$ *then* $x^*(R_n) \subset R_{n-k}$ *for* $n \geq k$ *and* $x^*(R_n) = 0$ *for* $n < k$.

b) *The following diagram commutes*

$$\begin{tikzcd} R \arrow[r, "m^*"] \arrow[rrr, bend right, "x^*"'] & R \otimes R \arrow[r, "\mathrm{id} \otimes \langle x|"] & R \otimes \mathbb{Z} \arrow[r, "\cong"] & R \end{tikzcd}$$

c) *If* $\rho \in R$ *is primitive then* ρ^* *is a derivation of* R *i.e.*

$$\rho^*(yz) = \rho^*(y) \cdot z + y \cdot \rho^*(z)$$

d) *If* $\rho \in R_n$ *is primitive,* $0 < k < n$*, and* $x \in R_k$ *then* $x^*(\rho) = 0$.

Properties of PSH algebras

We will only prove c).

Proof.

Let $\rho, y, z, u \in R$ with ρ primitive. Then $m^*(\rho) = \rho \otimes 1 + 1 \otimes \rho$ and

$$\begin{aligned}
\langle \rho^*(yz), u \rangle &= \langle yz, \rho u \rangle \\
&= \langle m(y \otimes z), \rho u \rangle \\
&= \langle y \otimes z, m^*(\rho u) \rangle \\
&= \langle y \otimes z, m^*(\rho) m^*(u) \rangle \\
&= \langle y \otimes z, (\rho \otimes 1) m^*(u) \rangle + \langle y \otimes z, (1 \otimes \rho) m^*(u) \rangle \\
&= \langle \rho^*(y) \otimes z, m^*(u) \rangle + \langle y \otimes \rho^*(z), m^*(u) \rangle \\
&= \langle m(\rho^*(y) \otimes z), u \rangle + \langle m(y \otimes \rho^*(z)), u \rangle \\
&= \langle \rho^*(y) \cdot z + y \cdot \rho^*(z), u \rangle
\end{aligned}$$

□

References

- Lang, S. (2002). *Algebra*.
- Zelevinsky, A. V. (1981). *Representations of finite classical groups: a Hopf algebra approach*
- Gruson, C., & Serganova, V. (2018). *A Journey Through Representation Theory*.