Hopf algebras and primitive elements

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Goal

State and prove Zelevinsky's decomposition theorem for PSH algebras.

Last time

- A PSH algebra R is a connected positive self-adjoint Hopf algebra over $\mathbb{Z}$.
- The set of basic elements, $\Omega = \Omega(R)$ is a $\mathbb{Z}$-basis consisting of homogeneous elements.
- The set of primitive elements (i.e. $m^*(x) = 1 \otimes x + x \otimes 1$) is denoted by P.
- x^* is the operator adjoint to multiplication by x.
- Given a T-group R, the set of positive elements in R is

$$R^+ = \{ \sum_{\omega \in \Omega(R)} m_\omega \cdot \omega \mid m_\omega \geq 0\}$$

In particular, $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid m \geq 0\}$.

Tensor products

If $\{R_\alpha \mid \alpha \in A\}$ is a family of PSH algebras then we define the tensor product

$$R = \bigotimes_{\alpha \in A} R_\alpha$$

to be the direct limit of the finite tensor products

$$\bigotimes_{\alpha \in S} R_\alpha$$

where S ranges over the finite subsets of A. Then R is a PSH algebra with

$$\Omega(R) = \coprod_S \Big(\prod_{\alpha \in S} \Omega(R_\alpha) \Big)$$

Decomposition theorem

Theorem 1

Any PSH algebra R decomposes into the tensor product of PSH algebras with only one basic primitive element.

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Specifically, let $\mathscr{C} = \Omega \cap P$. For any $\rho \in \mathscr{C}$ we set

$$\Omega(\rho) = \{\omega \in \Omega \mid \langle \omega, \rho^n \rangle \neq 0 \text{ for some } n \geq 0\}$$

and

$$R(\rho) = \bigoplus_{\omega \in \Omega(\rho)} \mathbb{Z} \cdot \omega$$

Then we will show that $R(\rho)$ is a PSH subalgebra of R with set of basic elements $\Omega(\rho)$, ρ is the unique basic primitive element of $R(\rho)$, and R as a PSH algebra is isomorphic to the tensor product $\bigotimes_{\rho \in \mathscr{C}} R(\rho)$.

Proving the decomposition theorem

Lemma 1

Let $\rho_1, \ldots, \rho_r, \rho'_1, \ldots, \rho'_s \in \mathscr{C}$, $\pi = \rho_1 \cdot \ldots \cdot \rho_r$, $\pi' = \rho'_1 \cdot \ldots \cdot \rho'_s$. *If* $(\rho_1, \ldots, \rho_r)$ *and* $(\rho'_1, \ldots, \rho'_s)$ *are not equal up to permutation, then*

$$\langle \pi, \pi' \rangle = 0$$

Proof. If $r = 1$ and $s = 1$ then $\langle \pi, \pi' \rangle = 0 \iff \rho_1 \neq \rho'_1$. If $r = 1$ and $s > 1$ then

$$\begin{aligned}
\langle \rho_1, \rho'_1 \ldots \rho'_s \rangle &= \langle m^*(\rho_1), \rho'_1 \otimes \rho'_2 \cdot \ldots \cdot \rho'_s \rangle \\
&= \langle \rho_1 \otimes 1 + 1 \otimes \rho_1, \rho'_1 \otimes \rho'_2 \cdot \ldots \cdot \rho'_s \rangle \\
&= \langle \rho_1, \rho'_1 \rangle \langle 1, \rho'_2 \cdot \ldots \cdot \rho'_s \rangle + \langle 1, \rho'_1 \rangle \langle \rho_1, \rho'_2 \cdot \ldots \cdot \rho'_s \rangle \\
&= 0
\end{aligned}$$

Proving the decomposition theorem

Lemma 1

Let $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_s \in \mathscr{C}$, $\pi = \rho_1 \cdot ... \cdot \rho_r$, $\pi' = \rho'_1 \cdot ... \cdot \rho'_s$. *If* $(\rho_1, ..., \rho_r)$ *and* $(\rho'_1, ..., \rho'_s)$ *are not equal up to permutation, then*

$$\langle \pi, \pi' \rangle = 0$$

Proof.

Suppose the lemma holds for all $r < q$.

Recall from last week that for $x \in R$ primitive,

$$x^*(yz) = x^*(y)z + yx^*(z)$$

so that

$$\rho_1^*(\rho'_1 \cdot ... \cdot \rho'_s) = \sum_{j=1}^{s} \rho'_1 \cdot ... \cdot \rho'_{j-1} \cdot \rho_1^*(\rho'_j) \cdot \rho'_{j+1} \cdot ... \cdot \rho'_s$$

Proving the decomposition theorem

Lemma 1

Let $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_s \in \mathscr{C}$, $\pi = \rho_1 \cdot ... \cdot \rho_r$, $\pi' = \rho'_1 \cdot ... \cdot \rho'_s$. *If* $(\rho_1, ..., \rho_r)$ *and* $(\rho'_1, ..., \rho'_s)$ *are not equal up to permutation, then*

$$\langle \pi, \pi' \rangle = 0$$

Proof.
Thus

$$\begin{aligned} \langle \pi, \pi' \rangle &= \langle \rho_2 ... \rho_q, \rho_1^*(\rho'_1 ... \rho'_s) \rangle \\ &= \langle \rho_2 ... \rho_q, \sum_{j=1}^{s} \rho'_1 ... \rho'_{j-1} \rho_1^*(\rho'_j) \rho'_{j+1} ... \rho'_s \rangle \end{aligned}$$

From last week's proposition, we have if $x \in R_k$ and $y \in R_n$ with $k \neq n$ and y primitive then $x^*(y) = 0$ and if $k = n$ then $x^*(y) = \langle x, y \rangle$. Thus if $\rho_1 \neq \rho'_j$ for all j then $\langle \pi, \pi' \rangle = 0$.

Proving the decomposition theorem

Lemma 1

Let $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_s \in \mathscr{C}$, $\pi = \rho_1 \cdot ... \cdot \rho_r$, $\pi' = \rho'_1 \cdot ... \cdot \rho'_s$. *If* $(\rho_1, ..., \rho_r)$ *and* $(\rho'_1, ..., \rho'_s)$ *are not equal up to permutation, then*

$$\langle \pi, \pi' \rangle = 0$$

Proof.

Otherwise, $\rho_1 = \rho'_j$ for k values of j. Then we have

$$\langle \pi, \pi' \rangle = k \langle \rho_2 ... \rho_q, \rho'_{i_1} ... \rho'_{i_{s-1}} \rangle$$

where the i_m are the new indexes for the ρ'_j after one (pick any) of the ρ'_j equal to ρ_1 is removed.
By the induction hypothesis, if $(\rho_2, ..., \rho_q)$ and $(\rho'_{i_1}, ..., \rho'_{i_{s-1}})$ are not equal up to permutation, then $\langle \pi, \pi' \rangle = 0$.
But if $(\rho_1, ..., \rho_r)$ and $(\rho'_1, ..., \rho'_s)$ are not equal up to permutation then neither are $(\rho_2, ..., \rho_q)$ and $(\rho'_{i_1}, ..., \rho'_{i_{s-1}})$, so we are done.
Q.E.D.

Proving the decomposition theorem

Denote by $S(\mathscr{C};\mathbb{Z}^+)$ the semigroup of functions $\varphi : \mathscr{C} \to \mathbb{Z}^+$ with finite support. For any $\varphi \in S(\mathscr{C};\mathbb{Z}^+)$ set

$$\pi_\varphi = \prod_{\rho \in \mathscr{C}} \rho^{\varphi(\rho)} \in R$$

We have $\rho \in R^+$, so by positivity of m, $\pi_\varphi \in R^+$. Denote by Ω_φ the set of irreducible constituents of π_φ (i.e. if $\pi_\varphi = \sum_{\omega \in \Omega} m_\omega \cdot \omega$ then Ω_φ is the set of elements ω such that $m_\omega > 0$). Equivalently,

$$\Omega_\varphi = \{\omega \in \Omega(R) \mid \langle \omega, \pi_\varphi \rangle \neq 0\}$$

Put

$$R_\varphi = \bigoplus_{\omega \in \Omega_\varphi} \mathbb{Z} \cdot \omega$$

Proving the decomposition theorem

Lemma 2

a) *The set* $\Omega = \Omega(R)$ *is a disjoint union* $\coprod_{\varphi \in S(\mathscr{C};\mathbb{Z}^+)} \Omega_\varphi$.

b) *The group* R *is graded by the semigroup* $S(\mathscr{C};\mathbb{Z}^+)$ *i.e.* $R = \bigoplus_{\varphi \in S(\mathscr{C};\mathbb{Z}^+)} R_\varphi$. *This grading is compatible with the Hopf structure, i.e.*

$$m(R_{\varphi'} \otimes R_{\varphi''}) \subset R_{\varphi'+\varphi''}$$

$$m^*(R_\varphi) \subset \bigoplus_{\varphi'+\varphi''=\varphi} (R_{\varphi'} \otimes R_{\varphi''})$$

Proof. Part a) Involves proving for all $\omega \in \Omega(R)$, there exists $\varphi \in S(\mathscr{C}, \mathbb{Z}^+)$ with $\langle \omega, \pi_\varphi \rangle > 0$ using induction on the degree of ω. Part b) follows from part a) and self-adjointedness.

Proving the decomposition theorem

Lemma 3

Let $\varphi, \varphi' \in S(\mathscr{C}, \mathbb{Z}^+)$ have disjoint supports. Then multiplication

$$m : R_\varphi \otimes R_{\varphi'} \to R_{\varphi+\varphi'}$$

establishes an isomorphism of T-groups. In other words, the elements $\omega \cdot \omega'$ where $\omega \in \Omega_\varphi$, $\omega' \in \Omega_{\varphi'}$, are basic, mutually distinct and any basic element in $\Omega_{\varphi+\varphi'}$ has such a form.

Proving the decomposition theorem

R.T.P.

$\omega_1 \cdot \omega_1'$ is basic and $\omega_1 \cdot \omega_1' \neq \omega_2 \cdot \omega_2'$ unless $\omega_1 = \omega_2$ and $\omega_1' = \omega_2'$.

Since $\omega_1 \in \Omega_\varphi$ and $\omega_1' \in \Omega_{\varphi'}$ we have $\langle \omega_1, \pi_\varphi \rangle \neq 0$ and $\langle \omega_1', \pi_{\varphi'} \rangle \neq 0$ which implies

$$\langle \omega_1 \omega_1', \pi_\varphi \pi_{\varphi'} \rangle \neq 0$$

Now it suffices to show

$$\langle \omega_1 \omega_1', \omega_2 \omega_2' \rangle = \delta_{\omega_1 \omega_2} \delta_{\omega_1' \omega_2'}$$

since this gives $\langle \omega_1 \omega_1', \omega_1 \omega_1' \rangle = 1$ so that $\omega_1 \omega_1' \in \Omega_{\varphi + \varphi'}$.
We have

$$\begin{aligned} \langle \omega_1 \omega_1', \omega_2 \omega_2' \rangle &= \langle m^*(\omega_1 \omega_1'), \omega_2 \otimes \omega_2' \rangle \\ &= \langle m^*(\omega_1) m^*(\omega_1'), \omega_2 \otimes \omega_2' \rangle \end{aligned}$$

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Now by Lemma 2 we have

$$m^*(\omega_1) \subset \bigoplus_{\varphi_0+\varphi_1=\varphi} R_{\varphi_0} \otimes R_{\varphi_1}$$

$$m^*(\omega_1') \subset \bigoplus_{\varphi_0'+\varphi_1'=\varphi'} R_{\varphi_0'} \otimes R_{\varphi_1'}$$

Thus

$$m^*(\omega_1)m^*(\omega_1') \subset \bigoplus_{\varphi_0,\varphi_1,\varphi_0',\varphi_1'} R_{\varphi_0+\varphi_0'} \otimes R_{\varphi_1+\varphi_1'}$$

Then the components of $m^*(\omega_1)m^*(\omega_1')$ which are not orthogonal to $\omega_2 \otimes \omega_2'$ satisfy $\varphi_0 + \varphi_0' = \varphi$ and $\varphi_1 + \varphi_1' = \varphi'$.

Proving the decomposition theorem

R.T.P.

$\omega_1 \cdot \omega_1'$ is basic and $\omega_1 \cdot \omega_1' \neq \omega_2 \cdot \omega_2'$ unless $\omega_1 = \omega_2$ and $\omega_1' = \omega_2'$.

Since φ and φ' have disjoint supports we solve the system of equations

$$\begin{cases} \varphi_0 + \varphi_1 = \varphi & \varphi_0' + \varphi_1' = \varphi' \\ \varphi_0 + \varphi_0' = \varphi & \varphi_1 + \varphi_1' = \varphi' \end{cases}$$

to get the unique solution $\varphi_0' = \varphi_1 = 0$, $\varphi_0 = \varphi$, $\varphi_1' = \varphi'$. Therefore

$$\begin{aligned} \langle \omega_1 \omega_1', \omega_2 \omega_2' \rangle &= \langle m^*(\omega_1) m^*(\omega_1'), \omega_2 \otimes \omega_2' \rangle \\ &= \langle (\omega_1 \otimes 1) \cdot (1 \otimes \omega_1'), \omega_2 \otimes \omega_2' \rangle \\ &= \langle \omega_1 \otimes \omega_1', \omega_2 \otimes \omega_2' \rangle \\ &= \langle \omega_1, \omega_2 \rangle \langle \omega_1', \omega_2' \rangle \\ &= \delta_{\omega_1 \omega_2} \delta_{\omega_1' \omega_2'} \end{aligned}$$

Proving the decomposition theorem

It remains to show that all elements of $\Omega_{\varphi+\varphi'}$ are of the form $\omega\omega'$ for $\omega \in \Omega_\varphi$, $\omega' \in \Omega_{\varphi'}$.
Write

$$\pi_\varphi = \sum m_i \omega_i, \qquad \pi_{\varphi'} = \sum n_j \omega'_j$$

Then

$$\pi_{\varphi+\varphi'} = \pi_\varphi \pi_{\varphi'} = \sum m_i n_j \omega_i \omega'_j$$

So the $\omega_i \omega'_j$ exhaust the irreducible constituents of $\pi_{\varphi+\varphi'}$.
Q.E.D.

Proving the decomposition theorem

Theorem 1

Any PSH-algebra R decomposes into the tensor product of PSH-algebras with only one basic primitive element.

Proof.

First note that $\pi_{n\chi_\rho} = \rho^n$ where $\chi_\rho \in S(\mathscr{C};\mathbb{Z}^+)$ is the characteristic function of the subset $\{\rho\} \subset \mathscr{C}$. Then the subspace $R(\rho) \subset R$ equals

$$R(\rho) = \bigoplus_{n \geq 0} R_{n\cdot\chi_\rho},$$

By Lemma 2.b, $R(\rho)$ is a PSH subalgebra of R (i.e. the restrictions of the morphisms are graded morphisms).

Clearly, any $\varphi \in S(\mathscr{C};\mathbb{Z}^+)$ has the form

$$\varphi = \sum_{\rho\in\mathscr{C}} n_\rho \cdot \chi_\rho, \quad n_\rho \in \mathbb{Z}^+$$

Proving the decomposition theorem

Theorem 1

Any PSH-algebra R decomposes into the tensor product of PSH-algebras with only one basic primitive element.

Proof.

Applying several times Lemma 3, we obtain an isomorphism of T-groups

$$m : \bigotimes_{\rho \in \mathscr{C}} R_{n_\rho \cdot \chi_\rho} \to R_\varphi$$

Since $R = \bigoplus_{\varphi \in S(\mathscr{C};\mathbb{Z}^+)} R_\varphi$, we repeatedly apply the distributive property of tensor products to obtain an isomorphism of PSH-algebras

$$\bigotimes_{\rho \in \mathscr{C}} R(\rho) \cong \bigotimes_{\rho \in \mathscr{C}} \bigoplus_{n \geq 0} R_{n \cdot \chi_\rho} \cong \bigoplus_{\varphi \in S(\mathscr{C},\mathbb{Z}^+)} R_\varphi \cong R$$

The last statement to be proved is that ρ is the unique basic primitive element in $R(\rho)$. This follows at once from Lemma 1.

References

- Zelevinsky, A. V. (1981). *Representations of finite classical groups: a Hopf algebra approach*
- Gruson, C., & Serganova, V. (2018). *A Journey Through Representation Theory*.
- Hoffman, P., Stange, K. E., & Woofi, C. (2003) *General Zelevinsky Algebras.*