Frobenius formula and its application to covering spaces of Riemann surfaces

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Frobenius number

Let $C_1, \ldots, C_k$ be conjugacy classes in a finite group G

$$\mathcal{N}(G; C_1, \ldots, C_k) := \#\{(c_1, \ldots, c_k) \in C_1 \times \cdots \times C_k | c_1 \cdots c_k = 1\}.$$

Since $c_i c_{i+1} = c_{i+1}(c_{i+1}^{-1} c_i c_{i+1})$, $\mathcal{N}(G; C_1, \ldots, C_k)$ is independent of the order of the arguments.

Theorem (Frobenius formula)

$$\mathcal{N}(G; C_1, \ldots, C_k) = \frac{|C_1| \cdots |C_k|}{|G|} \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_K)}{\chi(1)^{k-2}}$$

The sum is over all characters of irreducible representations of G.

- $k = 1$:

$$\mathcal{N}(G; C) = \#\{c \in C | c = 1\} = \begin{cases} 1, & C = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

Since $1 = g^{-1}hg \implies g = hg \implies h = 1$, $\{1\}$ is the conjugacy class including the identity element.
Recall that if A and B are square matrices, $tr(AB) = tr(BA)$.

$$\begin{aligned} \chi_\pi(g^{-1}hg) = tr(\pi(g^{-1}hg)) &= tr(\pi(g)^{-1}\pi(h)\pi(g)) \\ &= tr(\pi(g)^{-1}\pi(g)\pi(h)) \\ &= tr(\pi(h)) = \chi_\pi(h) \end{aligned}$$

So the character $\chi_\pi(g)$ only depends on the isomorphism class of representation π and the conjugacy class of g. Hence, $\chi_\pi(\{1\}) = \chi_\pi(1) = dim\pi$ and $\chi(1) = \overline{\chi(1)}$. By orthogonality relation,

$$\frac{|C|}{|G|} \sum_\chi \frac{\chi(C)}{\chi(1)^{-1}} = \frac{|C|}{|G|} \sum_\chi \chi(C)\chi(1) = \begin{cases} 1, & C = \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

- $k = 2: \quad \mathcal{N}(G; C_1, C_2) = \#\{(c_1, c_2) \in C_1 \times C_2 | c_1 c_2 = 1\}$, so $c_2 = c_1^{-1}$. If $a = g^{-1}bg$ and such that $a \in C_1$ and $a^{-1} \in C_2$, then $b \in C_1$ and $b^{-1} = ga^{-1}g^{-1} \in C_2$. Therefore,

$$\mathcal{N}(G; C_1, C_2) = \begin{cases} |C_1|, & C_2 = C_1^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

By orthogonality relation,

$$\frac{|C_1||C_2|}{|G|} \sum_{\chi} \chi(C_1)\chi(C_2) = \begin{cases} |C_1|, & C_2 = C_1^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

- $k = 3$: Let $(C_1, C_2, C_3) = (A, B, C^{-1})$ with $A, B, C \in \mathcal{C}$. Then

$$n_{AB}^{C} := \mathcal{N}(G; A, B, C^{-1}) = \#\{(a, b, c^{-1}) \in A \times B \times C^{-1} | abc^{-1} = 1\}$$
$$= \#\{(a, b) \in A \times B | ab \in C\}$$

$$\mathcal{Z}(\mathbb{Z}[G]) = < e_C = \sum_{c \in C} [c] | C \in \mathcal{C} >$$

Central elements $e_A = \sum_{a \in A} [a]$ and $e_B = \sum_{b \in B} [b]$ satisfy:

$$e_A e_B = \sum_{a \in A, b \in B} [ab] = \sum_{C} n_{AB}^{C} e_C.$$

Proof of Frobenius formula

To prove

$$\mathcal{N}(G; C_1, \ldots, C_k) = \frac{|C_1| \cdots |C_k|}{|G|} \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_K)}{\chi(1)^{k-2}},$$

we consider $G-$representation isomorphism $\mathbb{C}[G] \cong \bigoplus_{i \in I} End_{\mathbb{C}}(V_i) \cong \bigoplus_{i \in I} V_i^{\oplus n_i}$ where $n_i = dimV_i$, and compute the trace of multiplication by $e_{C_1} e_{C_2} \cdots e_{C_k}$ on both sides of it.

On the left handside: We know that $\mathbb{C}[G] =< [g] | g \in G >$, so $tr(\pi(1)) = |G|$. When $g \neq 1$ it acts as a permutation of base elements with no fixed points. Hence $tr(\pi(g)) = 0$.

$$tr(\pi(g), \mathbb{C}[G]) = \begin{cases} |G|, & g = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$e_{C_1} e_{C_2} \cdots e_{C_k} = \sum_{c_i \in C_i} [c_1 \cdots c_k]$$

$$tr(e_{C_1} e_{C_2} \cdots e_{C_k} : \mathbb{C}[G] \to \mathbb{C}[G]) = |G| \mathcal{N}(G; C_1, \ldots, C_k).$$

Proof of Frobenius formula (cont.)

On the right handside, since e_C is central, $\pi_i(e_C)$ commutes with every linear operator. Therefore by Schur's lemma, it acts on any irreducible representation V_i of G as scalar multiplication:

$$\forall i \in I \quad \pi_i(e_C) = \nu_{\pi_i}(C) \cdot Id.$$

$$\begin{aligned} |C|\chi_{\pi_i}(C) = \sum_{g \in C} \chi_{\pi_i}(g) &= tr(\pi_i(e_C), V_i) \\ &= tr(\nu_{\pi_i}(C) \cdot Id, V_i) \\ &= \nu_{\pi_i}(C) dim V_i \end{aligned}$$

Hence, $\nu_{\pi_i}(C) = \frac{|C|}{dim V_i} \chi_{\pi_i}(C) = \frac{\chi_{\pi_i}(C)}{\chi_{\pi_i}(1)} |C|$.

Proof of Frobenius formula (cont.)

Now assume $I = \{1, \ldots, m\}$ and as a warm-up consider the space $\bigoplus_{i=1}^{m} V_i$, the e_{C_j} acts as multiplication of

$$A_j = \begin{pmatrix} [\nu_{\pi_1}(C_j) \cdot I_{n_1 \times n_1}] & & 0 \\ & \ddots & \\ 0 & & [\nu_{\pi_m}(C_j) \times I_{n_m \times n_m}] \end{pmatrix}$$

Because the Schur's lemma assures that $Hom_G(V_i, V_j) = 0$.

$$e_{C_1} \cdots e_{C_k} = A_1 A_2 \cdots A_k = A : \bigoplus_{i=1}^{m} V_i \to \bigoplus_{i=1}^{m} V_i$$

Therefore, its trace equals

$$\sum_{i=1}^{m} \frac{\chi_{\pi_i}(C_1)\chi_{\pi_i}(C_2) \cdots \chi_{\pi_i}(C_k)}{\chi_{\pi_i}(1)^k} |C_1||C_2| \cdots |C_k|.$$

Proof of Frobenius formula (cont.)

Now consider the original space $\bigoplus_{i \in I} End_{\mathbb{C}}(V_i) \cong \bigoplus_{i \in I} V_i^{\oplus n_i}$. The dimension of $V_i^{\oplus n_i}$ is $n_i^2 = \chi_{\pi_i}(1)^2$. So by abuse of notation A, each diagonal block in A is $(n_i \times n_i) \times (n_i \times n_i)$ and its trace is

$$\frac{\chi_{\pi_i}(C_1)\chi_{\pi_i}(C_2)\cdots\chi_{\pi_i}(C_k)}{\chi_{\pi_i}(1)^k}|C_1||C_2|\cdots|C_k| \times \chi_{\pi_i}(1)^2.$$

Therefore,

$$|G|\mathcal{N}(G; C_1, \ldots, C_k) = \sum_{i=1}^{m} \frac{\chi_{\pi_i}(C_1)\chi_{\pi_i}(C_2)\cdots\chi_{\pi_i}(C_k)}{\chi_{\pi_i}(1)^{k-2}}|C_1||C_2|\cdots|C_k|.\square$$

Riemann surfaces

Consider the complex plain $\mathbb{C}$. It is a very simple example of Riemann surfaces. As a more interesting example we can mention 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$, also called Riemann sphere.

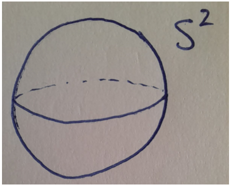


Figure: Riemann sphere.

Riemann surface

A Riemann surface is an orientable compact 2-dimensional topological manifold.

Riemann surfaces of higher genus

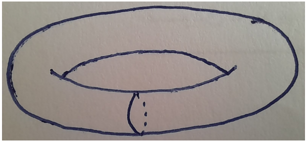


Figure: Torus is a Riemann surface of genus 1.

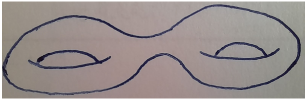


Figure: A Riemann surface of genus 2.

Covering space

Let Y be a topological space. A covering of Y is a space X together with a continuous surjective map $p : X \to Y$ such that the following holds:

$$\forall y \in Y \quad \exists (y \in) U \subset Y \quad p^{-1}(U) = \bigsqcup_{i \in I} V_i, \quad p|_{V_i} : V_i \xrightarrow[\text{cont. inverse}]{\text{bijective}} U.$$

p is called a covering map.

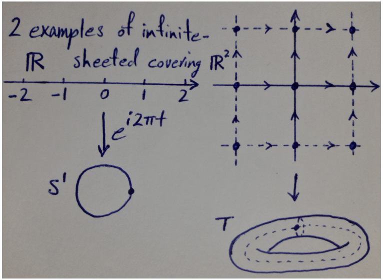


Figure: $\mathbb{R}$ and $\mathbb{R}^2$ as covering spaces of S^1 and torus.

Fundamental group

Let Y be the Riemann sphere with k points $P_1, \ldots, P_k$ removed.
By homotopy of a path we mean deforming it continuously and keeping its end points fixed.

Homotopy of paths

A homotopy of paths in Y is a family $\gamma_t : [0,1] \to Y, \quad 0 \le t \le 1$ such that

- The endpoints $\gamma_t(0)$ and $\gamma_t(1)$ are independent of s.
- The associated map $F : [0,1]^2 \to Y$ defined by $F(s,t) = \gamma_t(s)$ is continuous.

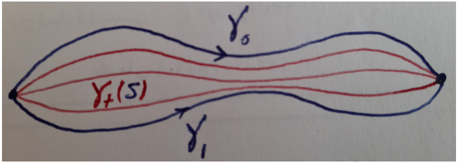


Figure: Paths homotopy.

Homotopy relation is an equivalence relation and we denote the class of γ by $[\gamma]$. Now consider the composition $\alpha \cdot \gamma$ defined by

$$\alpha \cdot \gamma(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

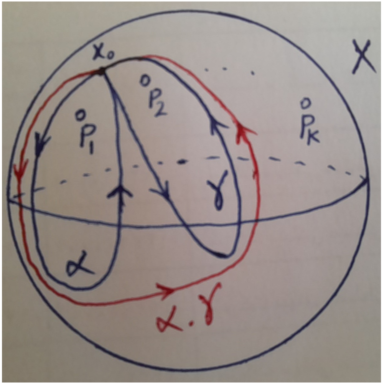
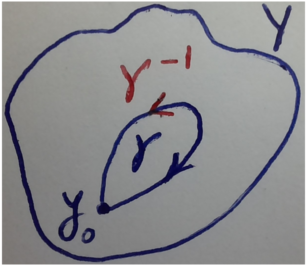


Figure: Paths composition.

Fundamental group

In particular, conisder loops with base point y_0. The set of all homotopy classes $[\gamma]$ forms a group with respect to composition $[\alpha] \cdot [\gamma] = [\alpha \cdot \gamma]$. This group, denoted by $\pi_1(Y, y_0)$, is called the fundamental group of Y at point y_0.

In this group the identity element is the constant loop and the inverse element $[\gamma]^{-1}$ is the same loop passed in the inverse direction.



A Riemann surface is called simply connected whenever its fundamental group is trivial.

Example of a simply connected surface

Riemann sphere is simply connected. $S^2 \setminus \{p\}$ is also simply connected because assume we omit north pole N. There is a continuous bijective map "stereographic projection" $f : S^2 \setminus \{N\} \xrightarrow{\cong} \mathbb{C}$ with continuous inverse.

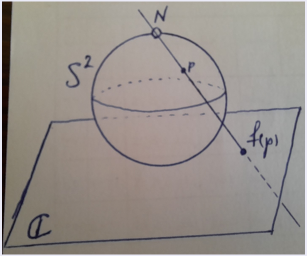


Figure: Stereographic projection.

$$f_\# : \pi_1(S^2 \setminus \{N\}) \xrightarrow{\cong} \pi_1(\mathbb{C})$$

Since every loop in the complex plane is contractible to a point, $\pi_1(S^2 \setminus \{N\}) = 0$.

Surfaces with non-trivial fundamental group

A loop γ in $\mathbb{C} \setminus \{P\}$ around the removed point can not be contracted into base point continuously. Therefore $[\gamma]$ is a non-unit element of fundamental group. Moreover, it is a generator, i.e.;
$\pi_1(\mathbb{C} \setminus \{P\}) =< [\gamma] >$.

$$\pi_1(\mathbb{C} \setminus \{P\}) \cong \mathbb{Z}.$$

Recall $Y = S^2 \setminus \{P_1, \ldots, P_k\}$,

$$\pi_1(Y) =< y_1, \ldots, y_k | y_1 \cdots y_k = 1 > .$$

Each generator is the class of loops around each removed point.
Remark: $\pi_1(Y) \cong< y_1, \ldots, y_{k-1} >$.

Fundamental group; a topological invariant

Theorem

Suppose that $p : X \to Y$ is a continuous map. Then it induces a homomrphism $p_{\#} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that $p_{\#}([\gamma]) = [p \circ \gamma]$.

Now consider the lift of paths and base point to the covering space.
Given any two homotopic paths in Y, are their pre-images homotopic?
This is true for covering spaces.

Lemma

Let (X, x_0) be a covering space of Y. Let α and β be paths in X with the same initial points. If $p\alpha \simeq p\beta$ then $\alpha \simeq \beta$.

Theorem

Let (X, x_0) be a covering space of (Y, y_0) such that $p(x_0) = y_0$. Then $p_{\#} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective.

Let $F = p^{-1}(y_0)$.

Monodromy group

For every $\gamma \in \pi_1(Y, y_0)$, it induces a bijection $g : F \to F$. Indeed:

- γ is a closed oriented curve in Y; therefore $p^{-1}(\gamma)$ consists of $|I|$ oriented curves in X.
- γ leads from y_0 to y_0; therefore each of the lifted curves in $p^{-1}(\gamma)$ leads from a point of F to a point of F, i.e.; a mapping $g : F \to F$.
- This mapping g is invertible since γ is invertible in $\pi_1(Y, y_0)$.

The correspondence $\gamma \mapsto g$ gives a group homomorphism from $\pi_1(Y, y_0)$ to $Bijections(F)$. The image G of this homomorphism is called the monodromy group of the covering.

Definition

Let (X_1, x_1) and (X_2, x_2) be covering spaces of Y. A homomorphism of (X_1, x_1) and (X_2, x_2) is a continuous map $\phi : X_1 \to X_2$ such that $p_2 \circ \phi = p_1$.

There is a bijective correspondence between Covering spaces of Y up to isomorphism and Subgroups of $\pi_1(Y, y_0)$. Let us fix $x_0 \in F$.

From a covering to a subgroup

Theorem

Let (X, x_0) be a covering space of (Y, y_0). Then the subgroups $p_\# \pi_1(X, x_0)$ for $x_0 \in p^{-1}(y_0)$ are conjugate subgroups of $\pi_1(Y, y_0)$.

In fact, each one of these subgroups are stabilizer of different base points (i.e.; includes paths composed of loop lifts, going from x_0 and returning to it) and

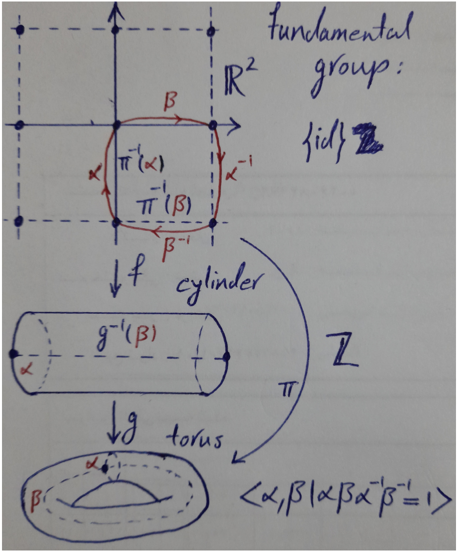
$$\forall x_0, x_1 \in F, \alpha \in H, \gamma \in K \quad \exists \beta \quad \beta \cdot x_0 = x_1 \Longrightarrow \alpha = \beta^{-1} \gamma \beta.$$

Moreover, the right cosets of H are in bijection with F:
$H\alpha = H\beta \Longleftrightarrow \alpha\beta^{-1} \in H \Longleftrightarrow$
both α and β send x_0 to the same element $x \in F$.
Therefore, $[\pi_1(Y, y_0) : H] = |F|$.

Normal covering

A covering is called normal if the corresponding subgroup $H \lhd \pi_1(Y, y_0)$ is normal.

In this case the monodromy group is isomorphic to the quotient group $\pi_1(Y, y_0)/H$.
A covering is called **universal** if $H = \{id\}$ i.e.; if the covering space is **simply connected**.



Let Σ_g be a Riemann surface of genus g, then

$$\pi_1(\Sigma_g) =< \alpha_1, \beta_1, \dots, \alpha_g, \beta_g | \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1 > .$$

From a subgroup to a covering

Given a topological space Y and a conjugacy class of subgroups of $\pi_1(Y, y_0)$, is there a covering space (X, x_0) such that $p_\# \pi_1(X, x_0)$ belongs to that conjugacy class?

Theorem

Let Y be a topological space which has a universal covering space. Then for any conjugacy class of subgroups of $\pi_1(Y, y_0)$ there exists a covering space (X, x_0) such that $p_\# \pi_1(X, x_0)$ belongs to the given conjugacy class.

Sketch of proof

Let $(\tilde{X}, \tilde{x}_0)$ be the universal covering. Since $\pi_1(Y, y_0)$ acts transitively and freely on F, $\pi_1(Y, y_0) \cong Mon(\tilde{X}, \tilde{x}_0)$. Let H be a nontrivial subgroup of $\pi_1(Y, y_0)$ belonging to the given class. It gives a subgroup K of $Mon(\tilde{X}, \tilde{x}_0)$. Let $X = \tilde{X}/K$, then $\tilde{X} \xrightarrow{r} X \xrightarrow{q} Y$ commutes with p.

Faithful action of a group on a set

Action of a group G on a set A is *faithful* when the corresponding permutative representation homomorphism $\phi : G \to Bijections(A)$ defined for all $g \in G$ by $\phi(g) = \sigma_g : A \to A \quad ; \quad \sigma_g(a) = ga$, is injective.

Let G be a finite group and $\rho : \pi_1(Y) \to G$ a group homomorphism. ρ is specified by its value on generators of the fundamental group.

$$1 = \rho(1) = \rho(y_1 \cdots y_k) = \rho(y_1) \cdots \rho(y_k)$$

Hence $\mathcal{N}(G; C_1, \ldots, C_k)$ simply counts the number of homomorphisms ρ with $\rho(y_i) \in C_i$ for each i.

Topological interpretation of Frobenius formula

If G acts faithfully on fibre F, then each such homomorphism corresponds to a normal covering X of Y with monodromy group G such that the permutation induced by $p^{-1}(\gamma)$, the lift of loop around P_i, belongs to the conjugacy class C_i.
In other words,

$$\mathcal{N}(G; C_1, \ldots, C_k) = \#\{p : X \to Y : \rho(y_i) \in C_i\}$$

This interpretation also illustrates the independence of $\mathcal{N}(G; C_1, \ldots, C_k)$ from the order of its arguments.

There is a natural question. **Can we generalize this to covering spaces of Riemann surfaces of arbitrary genus $g \geq 0$?**
We know that

$$\pi_1(\Sigma_g \setminus \{P_1, \ldots, P_k\}) = < \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k | \\ \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1 \cdots \gamma_k = 1 >$$

Define

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = \#\{(a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_k) \in \\ G^{2g} \times C_1 \times \cdots \times C_k : [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k = 1\}.$$

such that $[a, b] = aba^{-1}b^{-1}$. Then

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = \#\{p : X \to \Sigma_g \setminus \{P_1, \ldots, P_k\} | \\ \text{X is a normal covering with monodromy group } G \\ \text{such that monodromy of } \gamma_i \text{ lies in } C_i\}.$$

Theorem (generalized Frobenius formula)

With all the notations mentioned above, we have for all $g \geq 0$

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = |G|^{2g-1}|C_1| \cdots |C_k| \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}}.$$

Sketch of proof

Let $\mathcal{C}$ be the set of all conjugacy classes in G and a and a' be in the same conjugacy class A. Then there are $|G|/|A|$ elements $b \in G$ with $bab^{-1} = a'$. Hence

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = \sum_{A_1, \ldots, A_g \in \mathcal{C}} \frac{|G|}{A_1} \cdots \frac{|G|}{A_g} \times \mathcal{N}(G; A_1, A_1^{-1} \ldots, A_g, A_g^{-1}, C_1, \ldots, C_k)$$

Now we apply Frobenius theorem and othogonality relation to derive the generalized formula.

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Thank You!