Jordan-Hölder multiplicities of Verma modules with rational highest weight

3 Dec 2020

Melbourne Representation Theory Learning Seminar

Question: What are the multiplicities of $L(w\lambda)$ in $M(v\lambda)$ for $\lambda \in \mathfrak{h}^*$?

Gufang's convention: $M(v\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{v\lambda-\rho}$
(also Dragan's convention)

Gufang: $\lambda = -\rho$ (+ρ? I was confused by his notes...)

$\mathcal{O}_{-\rho}$	$\xleftrightarrow{BB}$	$\mathcal{D}_{\mathcal{B}}$-mod (N-eqvt)	$\xleftrightarrow{RH}$	$D^b_N(\mathcal{B})$	$\xleftrightarrow{\text{sheaf-fnct}}$	$\mathcal{H}$
M_w		$i_{w!}\,\mathcal{O}_{\mathcal{B}_w}$		$\underline{\mathbb{C}}_{\mathcal{B}_w}[\ell(w)]$		$q^{-\ell(w)/2} T_w$
L_w		$i_{w!*}\,\mathcal{O}_{\mathcal{B}_w}$		$IC(\overline{\mathcal{B}}_w)$		C_w

Here we have decomposition theorem

Answer: $\lambda = -\rho$

$[M_w, L_v] = P_{w_0 v, w_0 w}(1)$ ← Kazhdan-Lusztig polynomials

Okay, this answers our question for one value of λ. What about everything else?

integral λ: tensoring with line bundles immediately gives the answer

In $\mathcal{D}$-module categories have twist functor: For $\lambda \in P(\Sigma)$

$(\lambda): \mathcal{D}_{\mathcal{B}}\text{-mod} \longrightarrow \mathcal{D}_\lambda\text{-mod}$

$\mathcal{V} \longmapsto \mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{V}$

- equivalence of categories
- sends standards to standards, simples to simples

⇒ **Answer:** $\lambda \in P(\Sigma)$ (weight lattice)

$[M(w\lambda) : L(v\lambda)] = P_{w_0 v, w_0 w}(1)$

- What about general λ? ← Today's goal

Strategy: λ rational

$\mathcal{O}_\lambda \xleftrightarrow{BB + \text{sauce}} \mathcal{D}_{L^\circ}\text{-modules} + \text{monodromy} \xleftrightarrow{RH} D^b_{N\times k^*}(L^\circ) \xleftrightarrow{\text{sheaf-function}} KC_{L^\circ, n}$

- L°: line bundle w/ zero section removed
- $D^b_{N\times k^*}$: action depends on $n \geq 1$ "denominator"
- $KC_{L^\circ,n}$: collapses to $\mathcal{H}$ when $n=1$

Confession: It's not very clear to me how Lusztig's construction actually answers the multiplicity question, and his paper doesn't explain it. The strategy above is how I think it works very roughly, but perhaps after I explain the construction you all can help me understand better the translation to $\mathcal{O}_\lambda$

Today's Plan: ① Sketch of Lusztig's construction ② An alternate approach

Lusztig's Approach:

Warning: I'm very shaky on this machine, so I can only really sketch this construction... the details are a bit out of my reach

References:
- Lusztig, Characters of Reductive Groups over a Finite Field
  Ch 1 - Computation of local intersection cohomology of certain line bundles over a Schubert variety
- Kazhdan-Lusztig - Schubert varieties and Poincaré duality
- Lusztig-Vogan - Singularities of closures of K-orbits on a flag manifold

Set-up:

- 1st, a general construction: $H \curvearrowright X$ (H alg gp, X variety) all over $k = \overline{\mathbb{F}}_q$, $q = p^e$, fix $\ell \neq p$ prime

  - Choose $\mathbb{F}_q$-rational structures on H and X $\rightsquigarrow$ (geometric) Frobenius $F: H \longrightarrow H$, $F: X \longrightarrow X$

[Recall Dougal:

$X = X_0 \otimes_{\mathbb{F}_q} k$, $X_0 = \mathbb{F}_q$-variety (X_0: $\mathbb{F}_q$-rational structure) $\overset{\text{two}}{\underset{\text{maps}}{\rightsquigarrow}}$

- arithmetic Frobenius $X_0 \otimes k \longrightarrow X_0 \otimes k$, $x \otimes \lambda \longmapsto x \otimes \lambda^q$ — morphism of $\mathbb{F}_q$-varieties
- ★ geometric Frobenius (We will use this one →) $F_0 \otimes 1: X_0 \otimes k \longrightarrow X_0 \otimes k$ — morphism of $\overline{\mathbb{F}}_q$-varieties
  $F_0 \in \mathrm{End}(X_0)$ raises fncts on X_0 to q^{th} power

- ex: $X = \mathrm{spec}\, A$, $A = A_0 \otimes_{\mathbb{F}_q} k$

  geometric: $a \otimes \lambda \mapsto a^n \otimes \lambda$ raises coordinates to q^{th} power

  arithmetic: $a \otimes \lambda \mapsto a \otimes \lambda^n$ raises coefficients to q^{th} power]

- Build a category $\mathcal{C}_X$:
  - objects: $(\mathcal{F}, \phi)$ sheaves + sauce
    ($\mathcal{F}$: constructible, H-equivariant $\overline{\mathbb{Q}}_\ell$ sheaf over X)

    $\phi = \{ \phi_m : (F^m)^* \mathcal{F} \longrightarrow \mathcal{F} \}_{m \geq 1}$ (isomorphisms of sheaves; "sufficiently divisible" — I don't fully understand what this means, but later I think it will depend on n...)

    s.t. (i) ϕ_m is H-equivariant

    (ii) $\phi_{mm'} = (\phi_m)^{m'}$ when defined — note that if we have ϕ_1, we get all other m from this. However, we may not have ϕ_1...

  - morphisms: H-equt maps of sheaves commuting with all ϕ_m

  - eigenvalues of ϕ: Sauce = eigenvalues of Frobenius $\rightsquigarrow$ gives us "mixed" structure
    - For $x \in X$, have $\phi_m : \mathcal{F}_x \longrightarrow \mathcal{F}_x$ invertible linear transformation
    - Define eigenvalues of ϕ on $\mathcal{F}_x$ to be m^{th} roots of eigenvalues of ϕ_m
      (eigenvalues of ϕ: elts of $A := \overline{\mathbb{Q}}_\ell^\times / \text{roots of } 1$)

Next: Specialize this construction

$T \subset B \subset G$ — connected red. alg gp / $k = \overline{\mathbb{F}}_p$
($B \supset U$; T: torus w/ property $F(t) = t^p \; \forall t \in T$)

fix $\mathbb{F}_p$-rational structure (Frobenius $F: G \to G$)

$W = N_G(T)/T$

$R \supset R^+$

$G \curvearrowright \mathcal{B}$ = variety of Borel subgroups (conjugation)

$\mathcal{L} = \{G\text{-equt line bundles over } \mathcal{B}\}/\simeq \;\longleftrightarrow\; X(T)$ (group of characters)

$L_\lambda = \{(g,z) \in G \times k\}/B \;\longleftrightarrow\; \lambda$

where $b \cdot (g,z) = (gb^{-1}, \lambda(b) z)$

- For $L \in \mathcal{L}$, denote $L^\bullet = L$ w/o zero section, $\pi: L^\bullet \to \mathcal{B}$
- Apply general construction to

$H = U \times k^\times \curvearrowright L^\bullet$

action:
- Fix $n \geq 1$, $(n,p) = 1$
- $(u,x) \cdot (g,z) = (u \cdot g, x^n z)$ ← depends on n!

$\rightsquigarrow$ category $\mathcal{C}_{L^\bullet, n}$

Remarks:
- Depends on L and n
- $L \leftrightarrow \lambda \in X(T)$ "integral lambda"
- think: L = numerator, n = denominator — captures rational weights

Goal: Find some bases for the Grothendieck group $K\mathcal{C}_{L^\bullet,n}$

- orbits: $L^\bullet = \bigsqcup_{w \in W} \pi^{-1}(\mathcal{B}_w)$ ← Bruhat cells $\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w$
- isotropy groups: $\ell \in L^\bullet$

$\mathrm{Stab}_H \ell = U_\ell \times \mu_n \subset U \times k^\times$ (U_ℓ connected; μ_n = n^{th} roots of 1 in $k^\times$)

For $n \neq 1$, isotropy groups are disconnected!

- some sheaves: given $w \in W$, $\psi: \mu_n \to k^\times$ $\rightsquigarrow$

$\mathcal{X}^L_{w,\psi}$ = unique H-equt locally constant $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1 over $\pi^{-1}(\mathcal{B}_w) \subset L^\bullet$ s.t. μ_n acts on stalk by ψ

extend by zero to $L^\bullet$, call by same name

We're trying to find a basis for $K\mathcal{C}_{L^\bullet,n}$. Objects in $\mathcal{C}_{L^\bullet,n}$ are sheaves + sauce, so to turn $\mathcal{X}^L_{w,\psi}$'s into a basis we need to put a ϕ-structure on them.

- For $a \in A = \overline{\mathbb{Q}}_\ell^\times / \text{roots of } 1$, $\exists!$ Φ-structure on $\mathcal{X}^L_{w,\psi}$ s.t. Φ has eigenvalue a

(Parsing this:
- Φ_m is H-equt + $\pi^{-1}(\mathcal{B}_w)$ is an H-orbit, so each Φ_m is the same on all stalks
- $\mathcal{X}^L_{w,\psi}$ is rank 1, so on stalks, Φ_m is just multiplication by a # in $\overline{\mathbb{Q}}_\ell^\times$
- So if we fix $a \in A$, $\exists$ unique collection of Φ_m's s.t. this # is a^m.)

$\rightsquigarrow$ get collection of elts in $K\mathcal{C}_{L;n}$:

$$a\overline{\mathcal{X}}^L_{w,\psi} := \left[(\mathcal{X}^L_{w,\psi}, \phi(a)) \right] \in K\mathcal{C}_{L;n}$$

brackets mean iso class in Grothendieck group

($\phi(a)$: the unique Φ-structure w/ evalue $a \in A$)

- these form a $\mathbb{Z}$-basis of $K\mathcal{C}_{L;n}$

- Give $K\mathcal{C}_{L;n}$ structure of a $\mathbb{Z}[A]$-module by

$$a \cdot \overline{\mathcal{X}}^L_{w,\psi} = a\overline{\mathcal{X}}^L_{w,\psi}$$

- **The upshot:** Our first basis of $K\mathcal{C}_{L;n}$ (as a $\mathbb{Z}[A]$-module) is $\{ \overline{\mathcal{X}}^L_{w,\psi} \}$

- Next we construct a second basis.

  - Tool: Deligne, Goresky-MacPherson intersection cohomology

input: $\mathcal{X}^L_{w,\psi}$ $\longrightarrow$ $IC(\pi^{-1}(\overline{\mathcal{B}}_w), \mathcal{X}^L_{w,\psi}) \in D^b(\pi^{-1}(\overline{\mathcal{B}}_w))$

sheaf on $\pi^{-1}(\mathcal{B}_w) \subset \pi^{-1}(\overline{\mathcal{B}}_w)$ open dense

complex in derived category s.t. cohomologies

$\mathcal{H}^i(\pi^{-1}(\overline{\mathcal{B}}_w), \mathcal{X}^L_{w,\psi})$ constructible H-equt $\overline{\mathbb{Q}}_\ell$-sheaves on $\pi^{-1}(\overline{\mathcal{B}}_w)$

extend trivially to L°, get

$$\mathcal{H}^i(\mathcal{X}^L_{w,\psi}) \in \mathcal{C}_{L;n}$$

- constructible H-equt sheaf on L°
- Φ-structure inherited from $\mathcal{X}^L_{w,\psi}$

Define a second basis of $K\mathcal{C}_{L;n}$:

$$\left\{ \tilde{\mathcal{X}} := \sum_i (-1)^i \overline{\mathcal{H}}^i(\mathcal{X}^L_{w,\psi}) \right\}$$

($\overline{\mathcal{H}}^i(\mathcal{X}^L_{w,\psi})$: class in $K\mathcal{C}_{L;n}$ of object $\mathcal{H}^i(\mathcal{X}^L_{w,\psi})$)

We've constructed two basis of a Grothendieck group as a $\mathbb{Z}[A]$-module. You can probably guess where this is going. Our next step is to write one basis in terms of the other and analyze the coefficients.

Lemma 1: $\tilde{\mathfrak{X}}^L_{w,\psi}$ is uniquely characterised by two properties:

(1.a) $D(\tilde{\mathfrak{X}}^L_{w,\psi}) = [p]^{-(\ell(w)+1)}\, \tilde{\mathfrak{X}}^L_{w,\psi^{-1}}$ — here $[p] \in A$ is the image of $p \in \overline{\mathbb{Q}}_\ell^\times$ in A

(1.b) $\tilde{\mathfrak{X}}^L_{w,\psi} = \overline{\mathfrak{X}}^L_{w,\psi} + \underbrace{\mathbb{Z}[A]\text{-linear combo of } \overline{\mathfrak{X}}^L_{w',\psi}}$ for $w' < w$

coming from Weil conjectures? { coefficients are $\mathbb{Z}$-linear combos of $a \in A$ that are represented by algebraic #s in $\overline{\mathbb{Q}}_\ell^\times$ whose complex conjugates all have absolute value $\le p^{1/2(\ell(w)-\ell(w')-1)}$

- Here $D: K\mathcal{C}_{L^\cdot,n} \longrightarrow K\mathcal{C}_{L^\cdot,n}$ is Verdier duality
  - some properties:
    - D is $\mathbb{Z}$-linear
    - $D(ax) = aD(x)$ for $a \in A$, $x \in K\mathcal{C}_{L^\cdot,n}$
    - $D(\overline{\mathfrak{X}}^L_{w,\psi}) = \sum_{w' \le w} r_{w';w,\psi}\, \overline{\mathfrak{X}}^L_{w',\psi^{-1}}$ where $r_{w';w,\psi} \in \mathbb{Z}[A]$, $r_{w,w,\psi} = [p]^{\ell(w)-\ell(w')}$

[on $D^b_H(L^\cdot)$,
$D(\mathcal{F}) := R\mathrm{Hom}(\mathcal{F}, \text{dualising complex})$
complex of $\overline{\mathbb{Q}}_\ell$-sheaves on $L^\cdot$
$\rightsquigarrow$ on $K\mathcal{C}_{L^\cdot,n}$, $D(\mathcal{F}) = \sum (-1)^i D^i(\mathcal{F})$]

__Pause__. Examine case $n=1$

- $L_\lambda = \{(g,z) \in G \times k\}/B$ where $b\cdot(g,z) = (gb^{-1}, \lambda(b)z)$

- action $H \curvearrowright L^\cdot_\lambda$ is $(u,x)\cdot(g,z) = (u\cdot g, xz)$

- isotropy groups $\mathrm{Stab}_H \ell = U_1 \subset U \times k^\times$ connected $\longrightarrow$ only one local system on each orbit

- Over Bruhat cells, have trivial $k^\times$-bundle (true for any n...) $\pi^{-1}(\mathcal{B}_w) \simeq U/U_1 \times k^\times$

$$\pi^{-1}(\mathcal{B}_w) \simeq U/U_1 \times k^\times \;\xrightarrow{\ \pi\ }\; \mathcal{B}_w \simeq U/U_1$$

$k^\times$ acts freely on fibres (only true for $n=1$...)

$\Rightarrow$ H-equt sheaves on $\pi^{-1}(\mathcal{B}_w)$ $\overset{\sim}{\longleftrightarrow}$ U-equt sheaves on $\mathcal{B}_w$

$\Rightarrow$ Can work w/ U-equt $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathcal{B}$, $K\mathcal{C}_{L^\cdot,n}$ can be identified w/ Hecke algebra $\mathcal{H}$,

bases $\overline{\mathfrak{X}}^L_w \longleftrightarrow q^{\ell(w)/2} T_w$
$\tilde{\mathfrak{X}}^L_w \longleftrightarrow C_w$
(using Gufang's conventions...)

$\rightsquigarrow$ change of basis in (1.b) given by KL polys

__The upshot__: This construction collapses to the setting of U-equt $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathcal{B}$ when $n=1$.

Why is lemma true?

- $\tilde{\mathfrak{X}}^L_{w,\psi}$ satisfy (1.a) and (1.b) by def'n of $IC(-)$ and the "purity theorem of Gabber." I don't know what this theorem says. They cite BBD, but I don't know where in BBD to find it. I also don't speak French.

  Geordie's interpretation (see mathoverflow): if $\mathcal{F}$ is a pure perverse sheaf on an open subvariety $j: U \hookrightarrow X$ then so is $j_{!*}\mathcal{F}$.

  $\Rightarrow IC(\pi^{-1}(\mathcal{B}_w), \mathfrak{X}^L_{w,\psi})$ is pure, so Frobenius acts in a very special way on its cohomologies. This must imply the condition on the coefficients of $\tilde{\mathfrak{X}}^L_{w,\psi}$ in Lemma

- uniqueness straightforward and combinatorial

Lemma leads to **Question: What are the coefficients of $\tilde{\mathfrak{X}}^L_{w,n}$ in (1.b)?**

To answer, we need a

<u>Third basis of $\mathbb{k}e_{L,n}$:</u>

- W acts on $\mathfrak{L} = \{G\text{-eqvt line bundles over } \mathcal{B}\}/\simeq$:

  - G-orbits on $\mathcal{B} \times \mathcal{B}$ are $\mathcal{O}(w) = \{(B', B'') \in \mathcal{B}\times\mathcal{B} \mid B' \text{ and } B'' \text{ in rel. pos. } w\}$

    $(B', B'') \sim (B, {}^wB)$ (conjugate) where ${}^wB$ is: $W \overset{1:1}{\longleftrightarrow} \{\text{Borel subgroups containing } T\}$, $w \longleftrightarrow {}^wB$

  - Let $\mathfrak{L}_w = \left\{ \begin{array}{l} G\text{-eqvt line bundles} \\ \text{on } \mathcal{O}(w) \text{ w/ compatible} \\ \mathbb{F}_p\text{-rat'l structure} \end{array} \right\}$

  - Two projections
    $$\mathcal{B} \xleftarrow{pr_1} \mathcal{O}(w) \xrightarrow{pr_2} \mathcal{B} \quad \rightsquigarrow \quad \begin{array}{l} pr_1^*: \mathfrak{L} \longrightarrow \mathfrak{L}_w \\ pr_2^*: \mathfrak{L} \longrightarrow \mathfrak{L}_w \end{array} \quad \text{bijective}$$
    given by inverse image

  $\Rightarrow$ given $L \in \mathfrak{L}$, $\exists!$ elt ${}^wL \in \mathfrak{L}$ s.t. $\exists$ G-eqvt iso (over $\mathbb{F}_p$) $pr_1^*({}^wL) \xrightarrow{\sim} pr_2^* L$ (iso of line bundles over $\mathcal{O}(w)$)

  defines action of W: $w \cdot L = {}^wL$

(note: under the association $\mathfrak{L} \longleftrightarrow X(T)$ this aligns w/ the natural action of W on $X(T)$)

- Given L, n, define subgroup of W:

$$\left\{ \begin{array}{l} W_{L,n} = \{ w \in W \mid w\cdot L = L \otimes L_1^{\otimes n} \text{ for some } L_1 \in \mathfrak{L}_{root} \} \quad \text{Coxeter group} \\ R^\vee_{L,n} = \{ r^\vee \in R^\vee \mid r^\vee(L) \text{ is divisible by } n \} \rightsquigarrow R_{L,n}, R^+_{L,n}, R^-_{L,n}, S_{L,n} \end{array} \right.$$

($\mathfrak{L}_{root}$: subgroup gen'd by R)

if $n=1$, $W_{L,n} = W$, $R^\vee_{L,n} = R^\vee$

different from partial order inherited from W... (I think)

can show $\rightsquigarrow$ $R_{L,n}$ root system w/ Weyl group $W_{L,n}$, associate:

- $\underset{L,n}{\leq}$ partial order on $W_{L,n}$
- $\tilde{\ell} : W_{L,n} \longrightarrow \mathbb{N}$ length fnct

Some properties of $\underset{L,n}{\leq}$:

- Any coset $wW_{L,n} \in W/W_{L,n}$ has unique min length elt $w_1 \in W$
- If $v' \underset{L,n}{\leq} v$ and $w_1 \in w_1 W_{L,n}$ is min length, then $w_1 v' \leq w_1 v$ and $\ell(w_1 v) - \ell(w_1 v') \geq \tilde{\ell}(v) - \tilde{\ell}(v')$

With this we can define our third basis.

- Given $L, n, \psi : \mu_n \longrightarrow \overline{\mathbb{Q}}_\ell$, define

$$\widehat{\mathcal{X}}^L_{w,\psi} = \sum_{\substack{v' \in W_{L,n} \\ v' \underset{L,n}{\leq} v}} [p]^{\frac{1}{2}(\ell(w_1 v) - \tilde{\ell}(w_1 v') - \tilde{\ell}(v) + \tilde{\ell}(v'))} P_{v',v}([p]) \overline{\mathcal{X}}^L_{w_1 v', \psi}$$

here w_1 is min length in $wW_{L,n}$ and $w = w_1 v$

$P_{v',v}$: KL polys for $(W_{L,n}, S_{L,n})$

Main Lemma: $\widetilde{\mathcal{X}}^L_{w,\psi} = \widehat{\mathcal{X}}^L_{w,\psi}$

geometric basis = combinatorial basis

$\leftarrow$ This establishes that the coefficients in Lemma 1 are given by KL polys for the integral Weyl group $W_{L,n}$

Strategy:

- Show $\widehat{\mathcal{X}}^L_{w,\psi}$ satisfies (1.a) and (1.b)
- (1.b) follows from properties of KL polys
- The work is in showing (1.a) $\xrightarrow[\text{establish}]{\text{need to}}$ $D(\widehat{\mathcal{X}}^L_{w,\psi}) = [p]^{-(\ell(w)+1)} \widehat{\mathcal{X}}^L_{w,\psi^{-1}}$ (*)
- Approach:
  - Define push-pull operators $T_s, T_s' : K\mathcal{C}_{L,n} \longrightarrow K\mathcal{C}_{L,n}$

    $\mathcal{O}(s)$, $pr_1 : \mathcal{O}(s) \to \mathcal{B}$, $pr_2 : \mathcal{O}(s) \to \mathcal{B}$; ${}^sL \to \mathcal{B}$, $\mathcal{B} \leftarrow L$

    recall W-action gives H-equt iso

    $pr_1^*({}^sL) \overset{\gamma}{\underset{\sim}{\longrightarrow}} pr_2^* L$, $\pi_1 : pr_1^*({}^sL) \to {}^sL$, $\pi_2 : pr_2^* L \to L$

    $\rightsquigarrow$ $T_s(\mathcal{F}) = \sum (-1)^i R^i \pi_{1,!}(\gamma^* \pi_2^* \mathcal{F})$

    $T_s'(\mathcal{F}) = \sum (-1)^i R^i \pi_{1,*}(\gamma^* \pi_2^* \mathcal{F})$
  - Describe how they interact w/ D, $\widehat{\mathcal{X}}^L_{w,\psi}$ and $\overline{\mathcal{X}}^L_{w,\psi}$
    - $D T_s = [p]^{-1} T_s' D$
    - $T_s(\overline{\mathcal{X}}^L_{w,\psi}) = T_s'(\overline{\mathcal{X}}^L_{w,\psi}) = \begin{cases} \overline{\mathcal{X}}^{sL}_{ws,\psi} & ws > w \\ [p]\, \overline{\mathcal{X}}^{L'}_{ws,\psi} & ws < w \end{cases}$

- $T_s(\widehat{\mathfrak{X}}^{SL}_{ws,\psi}) = \widehat{\mathfrak{X}}^{L}_{w,\psi}$ if $ws < w$

- prove ⊛ by induction in $\ell(w)$ using properties of T_s established above
  - need to treat two cases $s \in W_{L,n}$ and $s \notin W_{L,n}$ separately □

- This establishes the main result of Lusztig by giving a topological interpretation for the multiplicities in the Jordan-Hölder series of a Verma module of rational highest weight.

★ Two big gaps in my understanding: ① Exactly how we pass this construction through RH and BB localization ② How we move from $k = \overline{\mathbb{F}_p}$ to $\mathbb{C}$ → [BBD §6 "De $\mathbb{F}$ a $\mathbb{C}$"]

- This machine of ℓ-adic sheaves is not my cup of tea. All of this should be able to be done using the machinery of mixed Hodge modules, but that's a world I am not yet competent in. However, to answer our original question, we actually don't need either ℓ-adic sheaves or MHM. In my last minutes of the talk, I'll sketch an argument using only $\mathcal{D}_\lambda$-modules. It's only a sketch because I haven't completely filled in the details yet. But if you ask again in a few months, hopefully Dragan and I will have written the whole thing down carefully...

A $\mathcal{D}$-module approach:

- BB localisation works for any $\lambda \in \mathfrak{h}^*$: $\mathcal{U}_\Theta\text{-mod} \underset{\Gamma}{\overset{\Delta_\lambda}{\rightleftarrows}} \mathcal{D}_\lambda\text{-mod}$ (~) $\qquad \Theta = W \cdot \lambda$

$\Rightarrow$ Computing $[\mathcal{I}(w,\lambda) : \mathcal{L}(v,\lambda)]$ answers our original question.

Same composition series:

Dual Verma $I(w\lambda) \longleftrightarrow \mathcal{I}(w,\lambda) := i_{w*}\mathcal{O}_{B_w}$

$M(w,\lambda) \longleftrightarrow \mathcal{M}(w,\lambda) := i_{w!}\mathcal{O}_{B_w}$

$L(w,\lambda) \longleftrightarrow \mathcal{L}(w,\lambda) := i_{w!*}\mathcal{O}_{B_w}$

- Define a map

$$\mathcal{D}_\lambda^{coh}\text{-mod} \xrightarrow{\nu} \text{free } \mathbb{Z}[q,q^{-1}]\text{-module w/ basis } \{\delta_w\}_{w\in W}$$

$$\mathcal{F} \longmapsto \sum_{w\in W}\sum_{m\in\mathbb{Z}} \dim_{\mathcal{O}}(R^m i_w^!(\mathcal{F}))\, q^m \delta_w$$

- Apply ν to $\mathcal{L}(w,\lambda)$, get polynomials

$$R_{wv} = \sum_{m\in\mathbb{Z}_+} \dim_{\mathcal{O}}(R^m i_v^!(\mathcal{L}(w,\lambda)))\, q^m$$

- Evaluation $\nu(-1)$ factors through Grothendieck group
- $\nu(\mathcal{I}(w,\lambda)) = \delta_w$

$\Rightarrow$ In $K\mathcal{D}_\lambda^{coh}\text{-mod}$, $[\mathcal{L}(w,\lambda)] = [\mathcal{I}(w,\lambda)] + \sum_{v<w} R_{wv}(-1)[\mathcal{I}(v,\lambda)]$

- **The upshot:** polynomials $R^{-1}_{w,v}$ determine multiplicities $[\mathcal{I}(w,\lambda):\mathcal{L}(v,\lambda)] = [M(w\lambda), L(v\lambda)]$

→ So what are these polynomials?

- If $\lambda = -\rho$, can show $R_{w,v}$ satisfy defining relations of Kazhdan-Lusztig polynomials
  i.e. $\mathcal{D}_\lambda = \mathcal{D}_X$, no twist → See Dragan's Localization notes

  * Key tool: Decomposition Theorem

- For arbitrary λ, we still have decomposition theorem (locally semisimple ⟺ semisimple for holonomic $\mathcal{D}$-modules)
  - claim: R_{wv} are KL polys for (W_λ, S_λ)
    ↑ integral Weyl group, gen'd by s_α for $\alpha \in R$ which pair integrally w/ λ

  steps: • Let $v, w \in W$. If $R^m i_v^! (\mathcal{L}(w,\lambda)) \neq 0$ for some $m \in \mathbb{Z}$, then $v \in w W_\lambda$

  ⤳ only nonzero $R_{w,v}$'s come from w,v in same right W_λ-coset

  (so the polys are at least the "right size")

  • proof by induction in $\ell_\lambda(w)$
    ↖ length fnct in (W_λ, S_λ)

    - tricky bit: induction step is subtle, must take into account both $s \in S_\lambda$ and $s \notin S_\lambda$, partial order given by (W_λ, S_λ) may differ from partial order given by (W, S).

  Stay tuned for more details...