Jordan-Hölder multiplicities of Verma modules with rational highest weight, part II

Melbourne Representation Theory Learning Seminar

Recall our question:

Question: What are the multiplicities of $L(w\lambda)$ in $M(v\lambda)$ for $\lambda \in \mathfrak{h}^*$?

Last time: via constructible sheaves on a line bundle over G/B (Lusztig)

Today: via $\mathcal{D}_\lambda$-modules

First- clarify conventions: Following set-up in Dragan Miličić's "localization" notes

$$\lambda \in \mathfrak{h}^* \rightsquigarrow \chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \xrightarrow[\text{HC hom}]{\varphi} S(\mathfrak{h}) \xrightarrow{\lambda-\rho} \mathbb{C} \qquad \rightsquigarrow \text{trivial infinitesimal character} = \chi_\rho$$

$$z = h + ux \longmapsto h \longmapsto h(\lambda - \rho)$$

$h \in \mathcal{U}(\mathfrak{h})$ $u \in \mathcal{U}(\mathfrak{g})$
$x \in \mathfrak{n}$

Why choose this convention?

NO DOT!

lemma: $\chi_\lambda = \chi_\mu \Leftrightarrow \exists w \in W$ s.t. $w\lambda = \mu$

(**proof:** $\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\varphi} S(\mathfrak{h}) \xrightarrow{\delta} S(\mathfrak{h})$

$h + ux \longmapsto h \longmapsto h : \mathfrak{h}^* \to \mathfrak{h}^*$

$\mu \longmapsto h(\mu - \rho)$

- composition is HC iso

$\delta \circ \varphi : \mathfrak{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$

$\delta \circ \varphi(z)(\mu) = \chi_\mu(z) = \chi_\lambda(z) = \varphi(z)(\lambda - \rho) = \delta \circ \varphi(z)(\lambda)$

$\Rightarrow$ have two maps $\mu, \lambda : S(\mathfrak{h}) \to \mathbb{C}$ which agree on $S(\mathfrak{h})$

$\Rightarrow \mu = w\lambda$ $\square$)

Verma modules:

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

- $\mathfrak{Z}(\mathfrak{g})$ acts by χ_λ on $M(\lambda)$

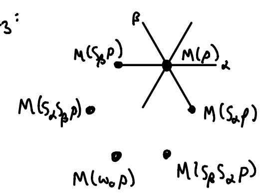
The upshot: My W-actions are never the dot action. (so center of W-action is 0) My Verma module $M(\lambda)$ has highest weight $\lambda - \rho$. The Vermas in the principal block of category $\mathcal{O}$ are $\{M(w\rho) \mid w \in W\}$.

Beilinson-Bernstein localisation

$\lambda \in \mathfrak{h}^*$ regular antidominant $\Theta = W\lambda$ (NO DOT)

$$\mathcal{U}_\Theta = \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \ker \chi_\lambda$$

($\mathfrak{sl}_3$:



)

$$\mathcal{U}_\Theta\text{-mod}^{f.g.} \underset{\Gamma}{\overset{\Delta_\lambda}{\rightleftarrows}} \mathcal{D}_\lambda\text{-mod}^{coh} \quad (\sim) \qquad \Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\Theta} V$$

Verma $M(w\lambda) \longleftrightarrow \mathcal{M}(w,\lambda) := i_{w!}\mathcal{O}_{X_w}$

dual Verma $I(w\lambda) \longleftrightarrow \mathcal{I}(w,\lambda) := i_{w*}\mathcal{O}_{X_w}$

simple $L(w\lambda) \longleftrightarrow \mathcal{L}(w,\lambda) := i_{w!*}\mathcal{O}_{X_w}$

($\mathcal{D}$-module functors: $X \xrightarrow{f} Y$ $\mathcal{D}$ TDO on Y

$f^* : \mathcal{D}\text{-mod} \longrightarrow \mathcal{D}^f\text{-mod}$

$\mathcal{V} \longmapsto \mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{D}} f^{-1}\mathcal{V}$

$f^! : D^b(\mathcal{D}\text{-mod}) \longrightarrow D^b(\mathcal{D}^f\text{-mod})$

$f^! := Lf^* \circ [\dim Y - \dim X]$)

· Because Δ_λ is an equivalence of categories for any $\lambda \in \mathfrak{h}^*$ antidominant, regular, computing

$$[\mathcal{M}(w,\lambda) : \mathcal{L}(v,\lambda)] = [\mathcal{I}(w,\lambda) : \mathcal{L}(v,\lambda)]$$

answers our initial question for all <u>regular</u> $\lambda \in \mathfrak{h}^*$

↑ Maybe Arun will tell us about the singular case some time?

- So how do we compute these multiplicities? Originally it was done by moving over to categories of perverse sheaves. But I claim that we don't actually need to leave the land of $\mathcal{D}_\lambda$-modules.

<u>Step 1</u>: Define some polynomials geometrically

$$\mathcal{H} := \bigoplus_{w \in W} \mathbb{Z}[q,q^{-1}]\,\delta_w$$ free $\mathbb{Z}[q,q^{-1}]$-module with basis W

· Define a map

$$(\mathcal{D}_\lambda, N)\text{-mod} \xrightarrow{\nu} \mathcal{H}$$

$$\mathcal{F} \longmapsto \sum_{w \in W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} \underbrace{(R^m i_w^! (\mathcal{F}))}_{} q^m \delta_w$$

this is an N-equt $\mathcal{D}_\lambda$-module on an N-orbit → decomposes into copies of $\mathcal{O}_{X_w}$

· Apply ν to $\mathcal{L}(w,\lambda)$, get some polynomials

$$\boxed{R^\lambda_{wv} := \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} (R^m i_v^! (\mathcal{L}(w,\lambda)) q^m}$$

<u>Step 2</u>: These polynomials determine multiplicities

· Evaluation $\nu(-1)$ factors through the Grothendieck group:

$$\begin{tikzcd}
(\mathcal{D}_\lambda, N)\text{-mod} \arrow[r, "\nu"] \arrow[dr] & \mathcal{H} \arrow[r, "ev_{-1}"] & \bigoplus_{w \in W} \mathbb{Z}\delta_w \\
& K((\mathcal{D}_\lambda, N)\text{-mod}) \arrow[ur, "\nu(-1)"'] &
\end{tikzcd}$$

pf) $\nu(\mathcal{F})(-1) = \sum_{w \in W} \sum_{m \in \mathbb{Z}} (-1)^m \dim_{\mathcal{O}} (R^m i_w^! (\mathcal{F}))\,\delta_w$

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to$ s.e.s.

$\overset{\text{l.e.s.}}{\rightsquigarrow} \cdots \to R^m i_w^! (\mathcal{F}_1) \to R^m i_w^! (\mathcal{F}_2) \to R^m i_w^! (\mathcal{F}_3) \to R^{m+1} i_w^! (\mathcal{F}_1) \to \cdots$

$$\Longrightarrow \sum_{m \in \mathbb{Z}} (-1)^m \dim_{\mathcal{O}} (R^m i_w^! (\mathcal{F}_2))\,\delta_w = \sum_{m \in \mathbb{Z}} (-1)^m \dim_{\mathcal{O}} (R^m i_w^! (\mathcal{F}_1))\,\delta_w + \sum_{m \in \mathbb{Z}} (-1)^m \dim_{\mathcal{O}} (R^m i_w^! (\mathcal{F}_3))\,\delta_w$$

□

- $\nu(\mathcal{I}(w,\lambda)) = \delta_w$

pf) $R^m i_v^! (i_{w*} \mathcal{O}_{X_w}) = \begin{cases} \mathcal{O}_{X_w} & m=0,\ v=w \\ 0 & \text{else} \end{cases}$

$\Rightarrow$ in the Grothendieck group,

$$[\mathcal{L}(w,\lambda)] = \sum_{v \in W} R^{\lambda}_{wv}(-1)\,[\mathcal{I}(v,\lambda)]$$

so $R^{-1}_{wv}(-1)$ determines multiplicity of $\mathcal{L}(w,\lambda)$ in $\mathcal{I}(v,\lambda)$!

step 3: What are these polynomials?

case 1: $\lambda = -\rho$

i.e. $\mathcal{D}_{-\rho} = \mathcal{D}_X$, no twist

Can show $R^{-\rho}_{wv}$ satisfy defining relations of KL polys

α simple root $\rightsquigarrow$ define $T_\alpha : \mathcal{H} \to \mathcal{H}$ by $T_\alpha(\delta_w) = \begin{cases} q\,\delta_w + \delta_{ws_\alpha} & ws_\alpha > w \\ q^{-1}\delta_w + \delta_{ws_\alpha} & ws_\alpha < w \end{cases}$

Theorem: $\exists!$ fnct $\varphi : W \to \mathcal{H}$ s.t.

(i) for $w \in W$, $\varphi(w) = \delta_w + \sum_{v<w} P_{wv}\,\delta_v$ $\quad$ KL polys $P_{wv} \in q\mathbb{Z}[q]$

(ii) if α and w are s.t. $\ell(ws_\alpha) = \ell(w) - 1$, then $\exists\, c_v \in \mathbb{Z}$ (depending on α and w) s.t.

$$T_\alpha(\varphi(ws_\alpha)) = \sum_{v \le w} c_v\,\varphi(v)$$

Key tool: Decomposition theorem

$\rightarrow$ see Dragon's localization notes for full argument

case 2: $\lambda \in \mathfrak{h}^*$ arbitrary (regular antidom)

- decomposition theorem still holds $\left(\begin{matrix}\text{locally}\\ \text{semisimple}\end{matrix} \Leftrightarrow \text{semisimple for holonomic } \mathcal{D}_X^\lambda\text{-modules}\right)$
- $R^m i_v^! (\mathcal{L}(w,\lambda)) = 0$ for all but certain $v \in W$, depending on λ

$\rightarrow$ Which ones?

- Let $W_\lambda = \{ w \in W \mid w\lambda - \lambda \in Q(\Sigma) \}$ (root lattice)
  - integral Weyl group
  - Weyl group of $\Sigma_\lambda =$ roots which pair integrally w/ λ
- Let $\Sigma^+_w = \Sigma^+ \cap (-w^{-1}(\Sigma^+))$ positive roots that $-w^{-1}$ keeps positive
- Lemma: $A_\lambda = \{ w \in W \mid \Sigma^+_w \cap \Sigma_\lambda = \emptyset \}$ is a cross-section of W/W_λ
- Lemma: If $R^m i_v^! (\mathcal{L}(w,\lambda)) \neq 0$, then $v \in wW_\lambda$

pf) · composition factors of $\mathcal{I}(w,\lambda)$ are $\mathcal{L}(v,\lambda)$ for $v \in wW_\lambda$ $\ell(v) \le \ell(w)$

$\rightarrow$ can take Γ and prove this using category $\mathcal{O}$ arguments

· then use induction on $\ell(w)$

$\rightsquigarrow$ If $s \in A_\lambda$, $w \in W_\lambda$,

$$\nu(\mathcal{L}(sw,\lambda)) = \sum_{v \in W_\lambda} \underbrace{\boxed{\sum_{m \in \mathbb{Z}_+} \dim_{\mathcal{O}} \left(R^m i_{sv}^! (\mathcal{L}(sw,\lambda))\right) q^m}}_{\text{“}R^{\lambda}_{sw,sv}\text{”}} \delta_{sv}$$

<u>Theorem</u>: (i) For any $s \in A_\lambda$, $R^\lambda_{sw,sv} = R^\lambda_{w,v}$

(/conjecture) (ii) $R^\lambda_{w,v}$ are the KL polys for (W_λ, S_λ)

<u>proof</u>: coming soon...