Mixed geometry over finite fields

Last time:

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{\mathbb{F}} & \longrightarrow & \operatorname{Spec} \mathbb{F}_q \\ pt/W\mathbb{R} & \longrightarrow & S^1 \end{array}$$

- X_0 - variety / $\mathbb{F}_q$
- $\mathcal{F}_0$ - constructible ℓ-adic sheaf on X_0
- $K_0^\bullet$ - complex of ℓ-adic sheaves w/ constructible cohomology

We defined

- $\mathcal{F}_0$ pointwise pure wt $w \in \mathbb{Z}$

  (condition on Frobenius eigenvalues at stalks)

- $\mathcal{F}_0$ mixed of wt $\leq w$

  $\iff$ $\mathcal{F}_0$ iterated ext^n of pointwise pure sheaves of wt $\leq w$.

- $K_0^\bullet$ mixed of wt $\leq w$

  $\iff$ $\mathcal{H}^i(K^\bullet)$ mixed of wt $\leq w+i$

- $K_0^\bullet$ pure of wt w

$\iff K_0^\bullet$ mixed of wt $\leq w$

and $\mathbb{D}(K_0^\bullet)$ mixed of wt $\leq -w$

Verdier dual

cone over curve of genus > 0

- Also saw example showing

pointwise pure $\not\Rightarrow$ pure

Today:

1. Pure $\not\Rightarrow$ pointwise pure

2. Some vague remarks about motives

E.g. [Deligne] f proper $\Rightarrow Rf_*$ preserves purity

1// Let $f : X_0 \to Y_0$ such that

- f proper
- X_0 smooth
- $\exists\, y \in Y_0$ such that $H^*(f^{-1}(y))$ is not pure.

$Y_0 \longrightarrow \operatorname{Spec} \mathbb{F}_q$

$H^n(f^{-1}(y), \overline{\mathbb{Q}}_\ell) \circlearrowleft F_q$

eigenvalues α are not all

<u>Weil q-numbers</u> of wt n.

$\alpha \in \overline{\mathbb{Q}}$ and all $\beta \in \mathbb{C}$ w/ some minimal poly have $|\beta| = q^{n/2}$.

Now look at $Rf_* \overline{\mathbb{Q}}_\ell$

pure of wt 0 on X

pure of wt 0 on Y

But $(R^n f_* \overline{\mathbb{Q}}_\ell)_y = H^n(f^{-1}(y), \overline{\mathbb{Q}}_\ell)$ not pure.

So $Rf_* \overline{\mathbb{Q}}_\ell$ not pointwise pure.

2// Motives over $\mathbb{F}_q$

Field k $\rightsquigarrow$ $\mathbb{Q}$-linear abelian category $Mot(k)$ "motives over k"

<u>Idea</u>: $Mot(k)$ is the universal $\mathbb{Q}$-linear abelian category with a functor

$$h : Var_k \longrightarrow Mot(k)$$

w/ structure + properties of a cohomology theory.

"Mixed" properties of cohomology of varieties over k are supposed to be shadows of structure in $Mot(k)$.

Thm (Milne 90's):

Assume the Tate Conjecture

Then

$$\mathrm{Mot}(\mathbb{F}_q)\otimes\mathbb{Q}_\ell \xrightarrow[\sim]{H^*_{et}(-,\mathbb{Q}_\ell)} \left\{\begin{array}{c}\text{semisimple } \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) \\ \text{reps over } \mathbb{Q}_\ell \\ \text{whose eigenvalues} \\ \text{are Weil } q\text{-numbers}\end{array}\right\}$$

Upshot: Motives over $\mathbb{F}_q$ "=" Mixed geometry

$\mathrm{Mot}(\mathbb{F}_q)$ semisimple

$\Updownarrow$

take numerical equivalence in defn

H^*_{et} factors through $\mathrm{Mot}(\mathbb{F}_q)$

$\Updownarrow$

take homological equivalence in defn.

Tate $\Rightarrow$ homological = numerical

Rmk: $\mathrm{Mot}(\mathbb{C}) \xrightarrow{H^*} \{\text{Mixed Hodge structures}\}$

not an equivalence???

Hierarchy:

Motives over $\mathbb{Q}$

$\swarrow$ $\searrow$

Motives over $\mathbb{C}$ ($\mathbb{R}$) $\qquad$ Motives over $\mathbb{Q}_p$

$\downarrow$ $\qquad$ $\downarrow$

Mixed Hodge modules $\qquad$ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-reps

"p-adic Hodge theory"

$\downarrow$

Motives over $\mathbb{F}_q$ "=" $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-rep mixed geometry

$\mathrm{Spec}\,\mathbb{F}_p$ $\qquad$ $\mathrm{Spec}\,\mathbb{Z}_p$ $\qquad$ $\mathrm{Spec}\,\mathbb{Q}_p$

$\mathrm{Mot}(k)$ is Tannakian,

$$1 \to \mathbb{C}^\times \to W(\mathbb{R}) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

unique nontrivial extension.

$$h(X) \otimes h(Y) = h(X \times Y).$$

$$\left\{\begin{array}{l}\text{semisimple } \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) \\ \text{reps over } \overline{\mathbb{Q}}_\ell \\ \text{whose eigenvalues} \\ \text{are Weil } q\text{-numbers}\end{array}\right\}$$

$$\parallel$$

$$\mathrm{Rep}\, \mathrm{Diag}(\text{Weil } q\text{-numbers})$$

$$\uparrow$$

$$\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) = \widehat{\mathbb{Z}}$$

$$\underbrace{\widehat{\mathbb{Z}}}_{\text{proalgebraic group}} = \lim_n \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\text{alg group}}$$

$$\mathcal{O}(\widehat{\mathbb{Z}}) = \operatorname{colim}_n \mathcal{O}(\mathbb{Z}/n\mathbb{Z})$$