

Mixed geometry over finite fields

Last time:

$$X \longrightarrow X_0$$

$$\downarrow \qquad \qquad \downarrow$$

$$\text{Spec } F \longrightarrow \text{Spec } F_q$$

$$\text{pt } w \in R \longrightarrow \mathcal{O}'$$

- X_0 - variety / F_q constructible
- \mathcal{F}_0 - ℓ -adic sheaf on X_0
- K_0^\bullet - complex of ℓ -adic sheaves w , constructible cohomology

We defined

- \mathcal{F}_0 pointwise pure wt $w \in \mathbb{Z}$
 (condition on Frobenius eigenvalues at stalks)
- \mathcal{F}_0 mixed of wt $\leq w$
 $\iff \mathcal{F}_0$ iterated extⁿ of pointwise pure sheaves of wt $\leq w$.
- K_0^\bullet mixed of wt $\leq w$
 $\iff H^i(K_0^\bullet)$ mixed of wt $\leq w+i$

• K_0 pure of wt w

$\Leftrightarrow K_0$ mixed of wt $\leq w$

and $D(K_0)$ mixed of wt $\leq -w$

\uparrow
Verdier dual / cone over curve
of genus > 0

• Also saw example showing

pointwise pure $\not\Rightarrow$ pure

Today:

1. Pure $\not\Rightarrow$ pointwise pure

2. Some vague remarks about
modules

E.g. [Deligne] f proper $\Rightarrow Rf_*$ preserves

$\Delta //$ let $f: X_0 \rightarrow Y_0$ such that
purity

• f proper

• X_0 smooth

• $\exists y \in Y_0$ such that $H^*(f^{-1}(y))$
is not pure.

$Y_0 \rightarrow \text{Spec } \mathbb{F}_q$

$$H^n(f^{-1}(y), \overline{\mathbb{Q}}_l) \rightarrow \mathbb{F}_q$$

eigenvalues are not all 1.

Weil q -numbers of wt n .

$\alpha \in \overline{\mathbb{Q}}_l$ and all $\beta \in \mathbb{C}$ w/ same
minimal poly have $|\beta| = q^{n/2}$.

Now look at $Rf_* \overline{\mathbb{Q}}_l$
pure of wt 0
on X
pure of wt 0 on Y

But $(Rf_* \overline{\mathbb{Q}}_l)_y = H^n(f^{-1}(y), \overline{\mathbb{Q}}_l)$
not pure.

So $Rf_* \overline{\mathbb{Q}}_l$ not pointwise pure.

2// Modules over \mathbb{F}_q

Field $k \rightsquigarrow \mathbb{Q}$ -linear
abelian category
 $\text{Mod}(k)$ "modules
over k "

Idea: $\text{Mod}(k) \rightarrow$ the universal
 \mathbb{Q} -linear abelian category
with a functor

$$h: \text{Var}_k \longrightarrow \text{Mod}(k)$$

w) structure + properties
of a cohomology theory.

"Mixed" properties of cohomology
of varieties over k are
supposed to be shadows
of structure in $\text{Mod}(k)$.

π^m (Milne 90's):

Assume the Tate Conjecture

Then

$$H_{\text{et}}^*(-, \mathbb{Q}_\ell)$$



$$\text{Mot}(\mathbb{F}_q) \otimes \mathbb{Q}_\ell \xrightarrow{\sim}$$

} Semisimple $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ reps over \mathbb{Q}_ℓ whose eigenvalues are Weil q -numbers }

Upshot: Motives " = " mixed geometry over \mathbb{F}_q

$\text{Mot}(\mathbb{F}_q)$ semisimple



take numerical equivalence in def^n

H_{et}^* factors through $\text{Mot}(\mathbb{F}_q)$



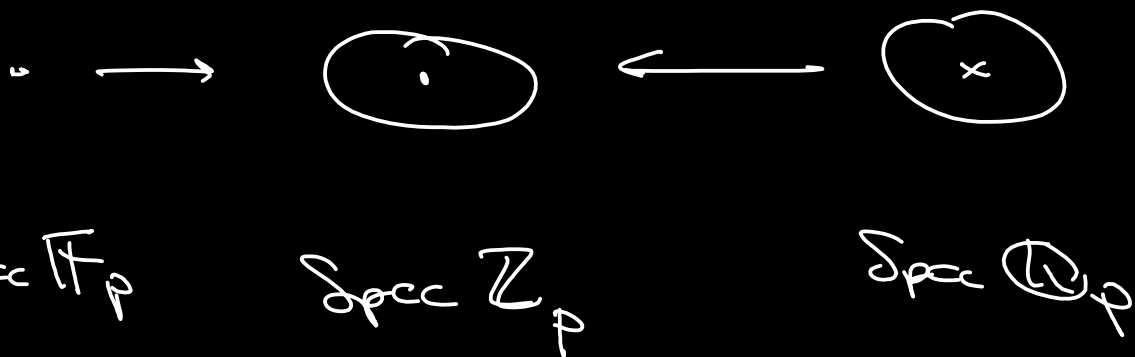
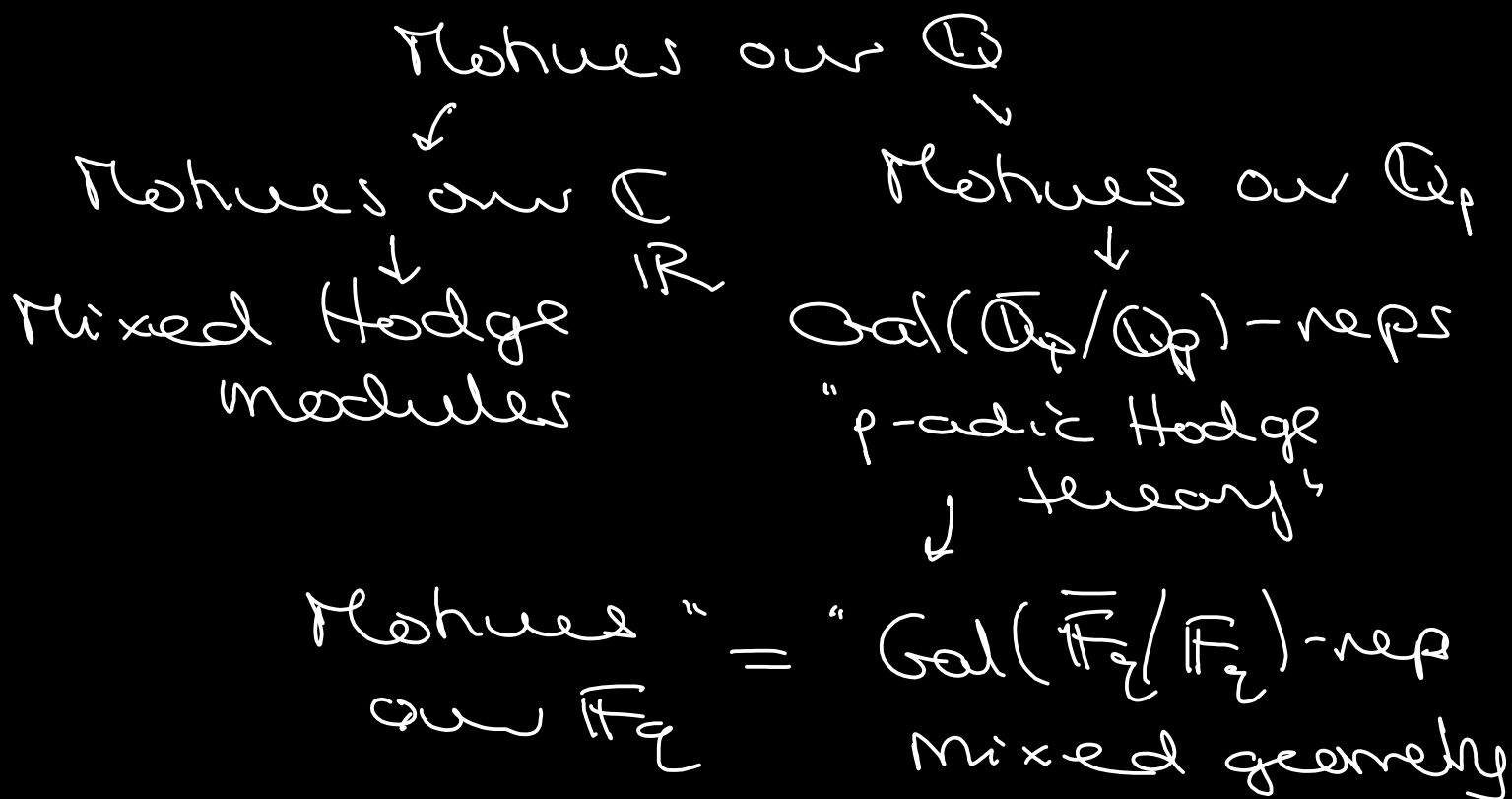
take homological equivalence in def^n .

Tate \Rightarrow homological = numerical

Link: $H_{\text{et}}^* \text{Mot}(\mathbb{F}) \rightarrow \left. \begin{array}{l} \text{Mixed Hodge} \\ \text{structures} \end{array} \right\}$

not an equivalence???

Hierarchy:



$\text{Mot}(k)$ is Tannakian,

$$1 \rightarrow \mathbb{Q}^x \rightarrow W(\mathbb{R}) \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow 1$$

unique nontrivial extension.

$$h(X) \otimes h(Y) = h(X \times Y).$$

Semisimple $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$
 reps over $\overline{\mathbb{Q}_\ell}$
 whose eigenvalues
 are Weil q -numbers

||

Rep Diag (Weil q -numbers)

$$\text{Gal}(\mathbb{F}/\mathbb{F}_q) = \widehat{\mathbb{Z}}$$

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

proalgebraic
group

alg group

$$\mathcal{O}(\widehat{\mathbb{Z}}) = \varprojlim_n \mathcal{O}(\mathbb{Z}/n\mathbb{Z})$$