Mixed geometry over finite fields

Main reference: [Deligne, Weil II]

Plan:

§1: (Pointwise) purity, mixedness

§2: Examples

~~§3: Motives over $\mathbb{F}_q$.~~

§1: Setup:

$\mathbb{F}_q$ - finite field w/ q elts

$\mathbb{F} = \overline{\mathbb{F}_q}$ alg closure

X_0 – scheme of finite type $/\mathbb{F}_q$

$X = X_0 \otimes \mathbb{F} = X_0 \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}$

ℓ prime s.t. $\ell \nmid q$.

$\rightsquigarrow D^b_c(X_0)$ = derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on X_0 w/ constructible cohomology.

$D^b_c(X)$ = — " — X — " —

$\mathcal{F}_0, K^\cdot_0 \in D^b_c(X_0) \rightsquigarrow \mathcal{F}, K^\cdot \in D^b_c(X)$
pullback.

<u>Frobenius</u>: $\varphi: \overline{\mathbb{F}} \xrightarrow{\sim} \overline{\mathbb{F}}$
$x \longmapsto x^q$

$F_q = \varphi^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$

$\Rightarrow F_q: X \longrightarrow X$ acting on $\overline{\mathbb{F}}$ factor (not geometric)

$\rightsquigarrow F_q \curvearrowright H^*(X) = H^*_{et}(X, \overline{\mathbb{Q}}_\ell)$

$X_0/k \rightsquigarrow X/\bar{k}$

Always have $\mathrm{Gal}(\bar{k}/k) \curvearrowright H(X)$.

$$\left\{ \begin{matrix} \overline{\mathbb{Q}}_\ell\text{-sheaves} \\ \text{on } \operatorname{Spec} \mathbb{F}_q \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \overline{\mathbb{Q}}_\ell\text{-vector spaces} \\ \text{w/ cts } \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) \\ \text{action} \end{matrix} \right\}$$

$$X_0 \xrightarrow{\pi} \operatorname{Spec} \mathbb{F}_q$$

$$R^i \pi_* \overline{\mathbb{Q}}_\ell \overset{\sim}{\longleftrightarrow} H^i(X, \overline{\mathbb{Q}}_\ell)$$

<u>Thm (Deligne)</u>: If X_0 is smooth + proper, then all eigenvalues α of F_q on $H^i(X, \overline{\mathbb{Q}}_\ell)$ are <u>Weil q-numbers</u> of wt i

i.e. $\alpha \in \overline{\mathbb{Q}}$ and for all $\beta \in \mathbb{C}$ with the same minimal polynomial,

$$|\beta| = q^{i/2}.$$

If $\mathcal{F}_0 \in D_c^b(X_0)$ sheaf, then

$$F_q : F_q^* \mathcal{F} \longrightarrow \mathcal{F}$$

If $x_0 \in X_0(\mathbb{F}_{q^n})$ then the image

$x \in X_0(\mathbb{F}) = X(\mathbb{F})$ of x_0 under $\mathbb{F}_{q^n} \hookrightarrow \mathbb{F}$

is fixed by F_q^n.

So get

$$F_{q^n} : \mathcal{F}_x \longrightarrow \mathcal{F}_x.$$

(Take $i^* \mathcal{F}_0$ on $\operatorname{Spec} \mathbb{F}_{q^n}$.)

<u>Def^n</u>: We say that

1) $\mathcal{F}_0$ is <u>pointwise pure of wt</u> w if for all x as above, the eigenvalues of F_{q^n} are Weil q^n-numbers of wt w.

2) $\mathcal{F}_0$ is <u>mixed</u> of wt $\leq w$ if $\mathcal{F}_0$ is an iterated ext^n of pointwise pure sheaves of wt $\leq w$.

3) For $K_0 \in D^b_c(X_0)$, say mixed of wt $\le w$ if $\mathcal{H}^i(K_0^\bullet)$ mixed of wt $\le w+i$ for all i.

Let $\omega_{X_0} = \pi^! \bar{\mathbb{Q}}_\ell$ be the dualizing complex of X_0, and

$\mathbb{D}(K_0^\bullet) = R\underline{Hom}(K_0^\bullet, \omega_{X_0})$.

(Verdier dual)

4) $K_0^\bullet$ is mixed of wt $\ge w$ if $\mathbb{D}(K_0^\bullet)$ is mixed of wt $\le -w$.

5) $K_0^\bullet$ is pure of wt w if it's mixed of wt $\le w$ and $\ge w$.

Rmk: Pointwise pure = $*$-pure

Thm (Deligne): $f: X_0 \to Y_0$ morphism of schemes $/ \mathbb{F}_q$, $K_0^\bullet \in D^b_c(X_0)$.

$K_0^\bullet$ mixed of wt $\le w \Rightarrow Rf_! K_0^\bullet$ mixed of wt $\le w$

Rmk: • $f^!$ lowers wts, f_* raises wts

f proper $\Rightarrow$ f_* preserves weights.

§2 : Examples

E.g. ⓪ $X_0 = \operatorname{Spec} \mathbb{F}_q$

Pointwise pure: all eigenvalues of wt w Weil q-numbers of wt w

Mixed: all eigenvalues $\geq w$ $(\leq w)$ Weil q-numbers of wt $\geq w$, $(\leq w)$

Pure = pointwise pure.

$\overline{\mathbb{Q}}_\ell$ - reps of $\hat{\mathbb{Z}}$ = $\overline{\mathbb{Q}}_\ell$ - reps of $\mathbb{Z}$ (gen by F_n) s.t. eigenvalues are in $\overline{\mathbb{Z}}_\ell^\times$.

for Weil q-number $\Rightarrow$ α algebraic integers.

Think: $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q) = \hat{\mathbb{Z}}$

$W(\overline{\mathbb{F}}/\mathbb{F}_q) = \mathbb{Z}$ ← Weil group.

① X_0 smooth $\Rightarrow$ cohomological shift raises wts

$$\omega_{X_0} = \overline{\mathbb{Q}}_\ell(\dim X_0)[2\dim X_0]$$

Tate twist (n) multiplies F_q-eigenvalues by q^{-n}.

<u>Upshot</u>: Can replace $\mathbb{D}(K_0)$ with $R\underline{Hom}(K_0, \overline{\mathbb{Q}}_\ell)$ to determine wts.

If $\mathcal{F}_0$ is lisse ($\Leftrightarrow$ local system) then

$\mathcal{F}$ pure of wt w $\Longleftrightarrow$ $\mathcal{F}$ pointwise pure of wt w.

Eg. ② Pointwise pure $\not\Rightarrow$ pure.

X_0 = cone over a curve C of genus g.

Constant sheaf $\overline{\mathbb{Q}}_\ell$ is pointwise pure of wt 0.

Question: Is $\mathbb{D}(\overline{\mathbb{Q}}_\ell) = \omega_{X_0}$ pointwise pure?

Away from sing pt x ✓

How about $i_x^* \omega_{X_0}$?

($i_x : \mathrm{Spec}\, \mathbb{F}_q \longrightarrow X_0$.)

Let
$$\begin{tikzcd} \tilde{X}_0 \arrow[d, "\pi"'] & C \arrow[l, "i_C"'] \arrow[d, "p"] \\ X_0 & \mathrm{Spec}\, \mathbb{F}_q \arrow[l, "i_x"'] \end{tikzcd}$$
be blowup at x.

We have

$$i_{x*} K^\circ \longrightarrow R\pi_* \omega_{\tilde{X}_0} \longrightarrow \omega_{X_0} \xrightarrow{[1]}$$

Apply $i_x^!$:

$$K^\circ \longrightarrow i_x^! R\pi_* \omega_{\tilde{X}_0} \longrightarrow i_x^! \omega_{X_0} \xrightarrow{[1]}$$

$$i_x^! \,\omega_{X_0} = \overline{\mathbb{Q}}_\ell \quad (\text{formal})$$

$$i_x^! \,R\pi_* \omega_{\tilde{X}_0} = Rp_* \, i_C^! \,\omega_{\tilde{X}_0} \quad \left(\text{proper base change}\right)$$

$$= Rp_* \,\omega_C$$

$$= R\Gamma(C, \overline{\mathbb{Q}}_\ell(1)[2])$$

<u>LES:</u>

$$0 \to H^{-2}(K^\cdot) \xrightarrow{\sim} H^0(C, \overline{\mathbb{Q}}_\ell(1)) \to 0,$$

$$\to H^{-1}(K^\cdot) \xrightarrow{\sim} H^1(C, \overline{\mathbb{Q}}_\ell(1)) \to 0,$$

$$\to \underbrace{H^0(K^\cdot)}_{\shortparallel\ 0} \to H^2(C, \overline{\mathbb{Q}}_\ell(1)) \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell \to 0$$

So:

$$H^i(K) = \begin{cases} H^{i+2}(C, \overline{\mathbb{Q}}_\ell(1)), & i = -2, -1 \\ 0, & \text{otherwise} \end{cases}$$

we also know

$$\omega_{\tilde{X}_0} = \overline{\mathbb{Q}}_\ell(2)[4], \text{ so}$$

$$i_x^* R\pi_* \omega_{\tilde{X}_0} = R\Gamma(C, \overline{\mathbb{Q}}_\ell(2))[4]$$

proper base change again

So have

$$K^\bullet \longrightarrow R\Gamma(C, \overline{\mathbb{Q}}_\ell(2))[4] \longrightarrow i_x^* \omega_{X_0}$$

Long exact sequence

wt -4

$$0 \longrightarrow H^0(C, \overline{\mathbb{Q}}_\ell(2)) \xrightarrow{\sim} H^{-4}(i_x^* \omega_{X_0}) \longrightarrow$$

wt -3

$$0 \longrightarrow H^1(C, \overline{\mathbb{Q}}_\ell(2)) \xrightarrow{\sim} H^{-3}(i_x^* \omega_{X_0}) \longrightarrow$$

c_1(normal bundle of C in $\tilde{X}_0$)

wt -1

$$H^0(C, \overline{\mathbb{Q}}_\ell(1)) \xrightarrow{\sim} H^2(C, \overline{\mathbb{Q}}_\ell(2)) \longrightarrow H^{-2}(i_x^* \omega_{X_0}) \longrightarrow$$

$$H^1(C, \overline{\mathbb{Q}}_\ell(1)) \longrightarrow 0$$

wrong weight!

Pointwise pure $\iff H^1(C, \overline{\mathbb{Q}}_\ell(1)) = 0$

$\iff g = 0.$

<u>Note</u>: $\overline{\mathbb{Q}}_\ell$ is pointwise $*$-pure but not pure!
(Mixed of wts ≤ 0 and ≥ -1)

③ Pure $\not\Rightarrow$ Pointwise pure.

Choose any $f : X \longrightarrow Y$ proper with X smooth and some $y \in Y$ such that $R\Gamma(f^{-1}(y), \overline{\mathbb{Q}}_\ell)$ not pure of wt 0.

singular

Then

$\overline{\mathbb{Q}}_{\ell,X}$ pure of wt 0
$\Longrightarrow$ $Rf_* \overline{\mathbb{Q}}_{\ell,X}$ pure of wt 0.
Deligne

But $i_y^* Rf_* \overline{\mathbb{Q}}_{\ell,X} = R\Gamma(f^{-1}(y), \overline{\mathbb{Q}}_\ell)$ <u>not</u> pure of wt 0, so $Rf_* \overline{\mathbb{Q}}_{\ell,X}$ not pointwise pure.

<u>Rmk</u>:

$*$-pure of wt w $\Rightarrow$ Mixed of wt $\leq w$

$!$-pure of wt w $\Rightarrow$ Mixed of wt $\geq w$