

Mixed geometry -

X space, Scheme $\xrightarrow{IH^*}$ Vector space / abelian gp

mixed Geometry $\rightsquigarrow$ ultimately give a motive

"Basic theory" perverse sheaves, D-modules

Put some extra structure

What we get on a point is "the extra structure".

Q: What is the extra structure?

A: Instead of v. space, we put repr. of a group.

Classical point of view, $G = \text{Galois group}$

X/F , F finite field, $G = \hat{Z} = \text{Gal}(\bar{F}/F)$.

F local field, $\mathbb{Q}_p$, $\text{Gal}(\bar{\mathbb{Q}_p}/\mathbb{Q}_p)$ & structure of this group?

Work over $\mathbb{C}$:

(2)

The complex version of mixed Hodge structures.

V space V , two filtrations W , $F^\cdot$, $\bar{F}^\cdot$
increasing decreasing

Induced filtrations $F^\cdot$ & $\bar{F}^\cdot$ are complementary on $\text{Gr}_n^W V$

ie: $\underbrace{F^p \oplus \bar{F}^{n-p}}_{\text{induced filtrations}} = \text{gr}_n^W V$

A vector space

$$V = \bigoplus V^{p,q}, \quad N: V^{p,q} \rightarrow \bigoplus_{\substack{a < p \\ b < q}} V^{a,b} \quad \text{linear map.}$$

To go back: $W_n = \bigoplus_{p+q=n} V^{p,q}$

$$F^p = \exp(N) \left(\bigoplus_{a \geq p} V^{a,b} \right), \quad \bar{F}^q = \exp(-N) \left(\bigoplus_{b \geq q} V^{a,b} \right).$$

Second description:

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$$\text{MHS} \Leftrightarrow \text{Rep } V \rtimes G_m \times G_m$$

$$\text{Lie}(V) = \left\{ \vartheta_{a,b} \mid \begin{matrix} a, b \\ \in \mathbb{Z} \end{matrix} \right\}$$

Solvable

$\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is solvable

Note: $\text{Cohom dim of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \text{ is } 2.$ } analogy
 $\text{Cohom. dim of MHS is } 1.$

$\text{Ind}_{G_m \times G_m}^G(\chi)$ is indecomposable pro-projective.

Claim:?

Any subbundle of a p.proj is proj?

$$H_{\mathcal{H}}^i(M) \stackrel{\text{def}}{=} \text{Ext}_{\text{MHS}}^i(\mathbb{C}(0), M)$$

only non-zero
in degrees 0 & 1.

(4)

 $X \xrightarrow{f} \text{pt} \quad \text{smooth (proper)}$

$$R\text{Hom}_{\text{MHM}_X}(f^*\mathbb{Q}(0), \mathcal{U}) = R\text{Hom}_{\text{MHM}}(\mathbb{Q}(0), Rf_*\mathcal{U})$$

↑

$f \text{ smooth} \Rightarrow f^*\mathbb{Q}(0)$ is of weight 0.

$$H^d(f^*\mathbb{Q}(0)) = 0, \quad \text{unless } d = \dim(X).$$

$\mathcal{O}_X \hookrightarrow$ weight $d = \dim X$ $[-d]$.

Rmk: DR: $\text{MHM}_X \rightarrow P(X)$

$$\mathcal{U} \mapsto \Omega_X \otimes \Omega[d \dim X]$$

Comment.

Gufang explained: how to get a graded enhancement of $\mathcal{O}_0$
from the mixed story.

It's not straight forward!

Jantzen Conjecture

⑤

We consider category $\mathcal{O}$ or $\mathcal{O}'$.

We write τ for the - of the Chevalley involution.

For an $\mathfrak{sl}_2$ -triple. $\tau(X_\alpha) = Y_\alpha$

$$\tau(Y_\alpha) = X_\alpha$$

$$\tau(h_\alpha) = h_\alpha$$

$M = M_\lambda$ Verma module of h.w λ .

To construct $M^\vee$, we dualize the weight spaces of M & let $\mathfrak{g}$ act on $M^\vee$.

$$\text{by } (uf)(v) = f(\tau(u)v).$$

Let v_+ be the h.w vector for M ,

$$v_+^\vee \quad \text{h.w vector for } M^\vee, \quad (v_+ v_+^\vee) = 1.$$

We get a natural map $I: M \rightarrow M^\vee$

$$v_+ \mapsto v_+^\vee.$$

⑥

If we go to geometry. $j! \xrightarrow{I} j_*$

The I gives us a contragredient pairing.

$$C: M \otimes M \longrightarrow \mathbb{C}$$

$$\text{s.t. } C(uv, v') = C(v, \tau(u)v') \text{ by}$$

$$C(v, v') = I(v)(v').$$

Note that:

↙ Call them $M, M^\vee$.

the underlying vector spaces of $M_\lambda \in M_\lambda^\vee$ can be identified for various λ , as these modules are just abstractly $\cong U(n_-) \vee$

$$I_\lambda: M \longrightarrow M^\vee, \quad \lambda \in \mathfrak{h}^*.$$

This map can have zeros.

Intertwining Operators

Rmk. Some people like it to have poles.

KL: they fix λ . want to understand how the standard modules & irreducibles are related. at λ .

(7)

Jantzen: Look at the neighborhood at λ !!!

Similar to looking at $\mathbb{F}_p \leftrightarrow \mathbb{Q}_p \supset \mathbb{Z}_p$. (•)

Let's consider two f.d vector spaces E & F .

$$A = \mathbb{C}[[t]]$$

$$K = \mathbb{C}(t)$$

$$It: A \otimes_{\mathbb{C}} E \hookrightarrow A \otimes_{\mathbb{C}} F \leftarrow \text{Free } A\text{-mod of rank } n.$$

$$\downarrow \otimes_A K$$

s.t.

get an isom.

We have a basis of $A \otimes_{\mathbb{C}} F$ $\{m_1, \dots, m_n\}$

s.t: $\{t^{\frac{d_1}{n_1}} m_1, \dots, t^{\frac{d_n}{n_n}} m_n\}$ is a basis of $A \otimes_{\mathbb{C}} E$. $d_1 | d_2 | \dots | d_n$.

(8)

Can define descending filtrations $F^\vee, E^\vee$ s.t.:

$$A \otimes_{\mathbb{C}} E^\vee = \langle m_i^\vee \mid d_i \geq v \rangle$$

$$A \otimes_{\mathbb{C}} F^\vee = \langle m_i \mid d_i \leq -v \rangle$$

} filtrations on
F&E

$$E^v / E^{v+1} \cong F^{-v} / F^{-v+1}$$

$$m \longmapsto t^{-m} I(m)$$

at $\mathcal{O}$.

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Jantzen Conjecture

M Verma module of $\mathfrak{h}$. weight λ .

$M^\vee$ contragredient dual

$M \rightarrow M^\vee \iff$ contragredient pairing on M .

$I(\lambda): M \rightarrow M^\vee$ (geom. $\tilde{j}! \rightarrow \tilde{j}_*$)

Goal: we will define increasing filtr $W'_i M$ $W'_i M^\vee$ as follows.

We can define a universal Verma module / $U(\mathfrak{h}) = R$.

In the construction of a Verma module, $\otimes$ everything by $U(\mathfrak{h})$

How does $\mathfrak{h}$ act?

$$\begin{array}{ccc} U(\mathfrak{h}) \otimes \mathfrak{h} & \rightarrow & U(\mathfrak{h}) \\ \downarrow \otimes & & \downarrow \\ 1 \otimes \mathfrak{h} & \xrightarrow{\quad} & \mathfrak{h} \end{array} \quad \text{h.c.}$$

$$M \text{ free mod } (\mathcal{U}(\mathfrak{g}))$$

$$M = U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R$$

(10)

To define the Jantzen filtration, we restrict $\lambda + \sum_{\alpha} a_{\alpha} \alpha$ (line in $\mathfrak{h}^*$)
any dominant wt

$$I(0): E \rightarrow F \quad E, F \text{ s.d. dim } l_{\mathbb{C}}$$

$$I(t): A \otimes_{\mathbb{C}} E \rightarrow A \otimes_{\mathbb{C}} F \quad (\text{two lattices}), \text{ where } A = \mathbb{C}[[t]], \quad K = \mathbb{C}((t)).$$

$$\text{s.t. } K \otimes_A () \xrightarrow{\text{isom}} K \otimes_A () \quad \text{get an isom.}$$

$$E_r := \{ v \in A \otimes_{\mathbb{C}} E \mid t^r I(t) v \in A \otimes_{\mathbb{C}} F \}$$

The same definition for F_r using $I(t)^t: K \otimes_{\mathbb{C}} F \rightarrow K \otimes_{\mathbb{C}} E$.

We get:

$$\dots E_{-2} \subseteq E_{-1} \subseteq E_0 = E \quad (\text{evaluate at } 0)$$

$$0 \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq F_k \subseteq F \quad \text{mod } t$$

Pf: $\exists$ basis $m_1, \dots, m_n$ of $A \otimes_{\mathbb{C}} F$ s.t. (11)

$m'_i = t^{d_i} m_i$ is a basis of $A \otimes_{\mathbb{C}} E$, $d_1 \mid d_2 \mid \dots \mid d_n$. through the image of I

$$E_r = \langle \{ m'_k(0) \mid d_k \geq -r \} \rangle$$

$$F_r = \langle \{ m'_k(0) \mid d_k \leq r \} \rangle$$

$$E_{-r} / E_{-r-1} \longrightarrow F_r / F_{r-1}$$

$$v \longmapsto t^{-r} I(t) v(0).$$

□

$M \longrightarrow M^\vee$ restriction $\pi + t\mathfrak{s}$ wt space and complete at 0.

$\Rightarrow$ Jantzen filtration on M_π & $M_\pi^\vee$.

The $w'_i M$, $w'_i M^\vee$ are $\mathcal{U}(\mathfrak{g})$ -submodules.

Fact: $w_{-r} M / w_{-r-1} M \cong w_r' M^V / w_{r-1}' M^V$

Let's specialize to $n=0$. Can work in $\mathcal{O}_0, \mathcal{O}_0'$.

For $y, w \in W$, we have polys $P_{y,w}(t)$ defined by Gufang.

Jantzen Conj: $n=0$.

Ting's indexing, $w \in W \rightsquigarrow M(w, 0) \xleftrightarrow{M_w} j_w! \mathbb{C}_{X_w \cup \{0\}} []$

$$X_e = pt$$

kL Conj:

$$M_y = \sum_{w \leq y} P_{w,y}(1) L_w.$$

Jantzen:

$$P_{w,y}(t) = \sum_i [Gr_{-i} M_y : L_w] t^i$$

$\text{gr}_i^{W'} M \xleftarrow{\text{non-deg}} \text{contragredient pairing.}$

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Rmk: The filtration W' is the weight filtration. (not obvious).

Monodromy wt filtration: (BB)

Let Q be an object in any abelian category and $S \in \text{End}(Q)$ nilpotent

Then, there exists a unique filtration μ on Q st:

$$S: \mu_i Q \rightarrow \mu_{i-2} Q.$$

and

$$S^i \text{ induces an isom: } \text{Gr}_i^\mu(Q) \xrightarrow{\sim} \text{Gr}_{-i}^\mu(Q)$$

Rmk: $\mu_i Q = \sum_{k-j=i} \ker(S^{k+1}) \cap \text{Im}(S^j).$

$$P_i = \ker(S: Gr_i \mathbb{Q} \rightarrow Gr_{i-2} \mathbb{Q}) \quad \text{primitive part.}$$

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$$Gr_*^u \mathbb{Q} = \bigoplus_{j \leq 0} P_j \otimes \mathbb{Z}[S] / S^{-j+1} \quad \text{graded mod}$$

$$\deg(P_i) = -i$$

$$\deg(S) = -2.$$

Structure:

$$\begin{array}{ccc} & P_{-1} & P_{-2} \\ & \downarrow S & \\ P_0 & & P_{-2} S \\ & \downarrow S & \\ & P_{-1} S & \\ & & P_{-2} S^2 \end{array}$$

$$\begin{array}{ccccccc} 0 & \rightarrow & \ker(S) & \rightarrow & \mathbb{Q} & \xrightarrow{S} & \mathbb{Q} \rightarrow \operatorname{coker}(S) \rightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \overline{J}! & & & & \overline{J}^* \end{array}$$

Filtration on $\mathbb{Q}$ induces a filtration on $\ker(S)$ & $\operatorname{coker}(S)$.

$$\begin{cases} W_i' J_! = J_! \cap \operatorname{Im}(s^{-i}) & i \leq 0 \quad \text{Stabilizer} \leftarrow \text{Negative degrees} \\ W_i' J_* = (\ker s^i + \operatorname{Im} s) / \operatorname{Im} s & \leftarrow \text{Filtration in pos. degrees} \end{cases}$$

(This is the weight filtration.)

Strict exact sequences

$$(*) \quad 0 \rightarrow (J_!, w_i') \rightarrow (Q, u) \rightarrow (\bar{Q}, u_{i-1}) \rightarrow 0$$

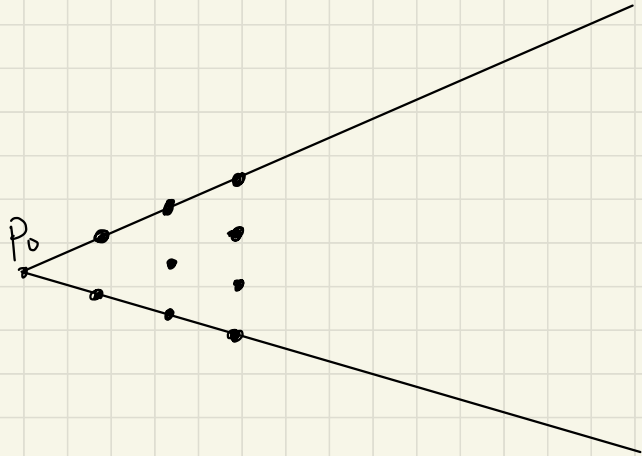
$$0 \rightarrow (\bar{Q}, u_{i+1}) \rightarrow (Q, u) \rightarrow (J_*, w_i') \rightarrow 0$$

where $\bar{Q} := Q / \ker(s) \hookrightarrow \bar{S}$

Note: $\bar{Q} \cong \operatorname{Im}(s)$.

$$\text{Gr } J! = \bigoplus_{j \leq 0} P_j s^{-j}$$

$$\text{Gr } J_* = \bigoplus_{j \leq 0} P_j.$$



Apply Gr to (*):

$$0 \rightarrow \bigoplus_{j \leq 0} P_j s^{-j} \xrightarrow{\quad} \bigoplus_{j \leq 0} P_j \otimes \mathbb{Z}[s] / s^{-j+1} \rightarrow \bigoplus_{j \leq 0} P_j \mathbb{Z}[s] / s^{-j} \rightarrow 0.$$

$\parallel$
 $\text{Gr } J!$

$$Q = \bigcup f$$

$$f: X \rightarrow \mathbb{C}$$

$$U \hookrightarrow X \xrightarrow{i} F$$

$f^{-1}(\mathbb{C}^*)$ closed.

$$P(u) \xrightarrow{\text{pass}} P(x)$$

Nearby cycle construction

$$j! \begin{matrix} \nearrow j! * \\ \searrow (?) \end{matrix} \begin{matrix} \nwarrow j_x \\ \nearrow \end{matrix}$$



when ! & * collide.

see: $\square_f$

$$\begin{array}{ccc} u & x & F \\ \parallel & \parallel & \parallel \\ X_w & \xleftrightarrow{\quad} \overline{X}_w & \xleftrightarrow{\quad} 2\overline{X}_w \end{array} \leftarrow \begin{array}{c} \square_f \\ \text{a principal divisor} \end{array}$$

$S = \log.$ unipotent monodromy.

Inputs:

kL , mixed
geometry



Koszul duality
(Statement)



more things

12 Nov 2020:

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Elementary to define the Jantzen filtration on $M_n \in M_n^\vee$.

Next step is to show it is the weight filtration (shifted)

We use mixed geometry. Gufang chose to use Hodge modules.

We have the category $\text{MHM}(X)$

Recall the objects are $(M, F.M, W.M)$
 $\begin{array}{ccccc} & \uparrow & & & \uparrow \\ & \mathcal{P} & & \mathcal{F} & \mathcal{R} \\ & D_X\text{-mod} & & D_X\text{-filtration} & \text{by } D_X\text{-modules.} \end{array}$

$$\cdot D_X(n) F_m M \subseteq F_{m+n} M$$

$$\cdot D_X W_m M \subseteq W_m M$$

Recall: M hol regular.

The quotient $W_r M / W_{r-1} M$ is of weight r . It is semi-simple.

If $\text{Gr}_r^W M \neq 0$, we say that M is of weight r

$$M \text{ has wt} \geq r \iff \text{Gr}_t^W M = 0 \text{ for } t < r$$

M has $\text{wt} \leq r \iff \text{Gr}_i^W M = 0$ for $i > r$.

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If M is a complex, then M has $\text{wt} \leq w \iff H^k(M)$ is of $\text{wt} \leq k+w$.
 $\geq w \iff$ — " — $\geq k+w$

We have 6 functor formalism

f_* $f^!$ increase wts

$f_!$ f^* decrease wts.

Socle filtration of M . $\leftarrow$ any arbitrary cat

$S_0 M =$ maximal semi-simple subobj.

$S_i M =$ pullback of $S_0(M/S_{i-1}M)$.

Assume M is a mixed Hodge module, M^{real} forget the mixed Hodge structure

They both have their socle filtrations.

These socle filtrations are different in general as one can see already when X is a point.

Def: We say that M is partially pure, $*$ -pure or $!$ -pure of weight w , if $i_x^* M$ is pure of wt w for all $x \in X$,
 $i_x: \{x\} \hookrightarrow X$.

if $i_x^! M$ ————— " —————

Very strong condition.

It's true for pure objects on flag manifolds w.r.t. N - or K -orbits.

Lemma (Beilinson - Bernstein)

$Y \stackrel{i}{\hookrightarrow} X$ locally closed, M pure of wt w on Y . Then for any

$N \subseteq \mathcal{H}^0(i_x^* M)$ in MHM s.t. its subquotients are all $!$ -pointwise pure.

We have: $S_i N = W_{w+i} N$.

Furthermore, the mixed and non-mixed socle filtrations coincide.

Nearby cycle Construction:

Consider X , $f: X \rightarrow \mathbb{C}$
 φ
 smooth

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{j} & X & \xrightarrow{i} & F \\ \text{"} & & & & \text{"} \\ f^{-1}(\mathbb{C}^*) & & & & f^{-1}(b) \end{array}$$

Consider a holonomic module M on $\mathcal{U}$. We will analyze $j_* M$ & $j_! M$.

We will work locally near a point in F .

Thus, we can assume that $\mathcal{U}$ is Stein. (affine in alg cat)

Hence, M is generated by a $\mathcal{O}_{\mathcal{U}}$ -coherent submodule M_0 .

A serious input into $\mathcal{D}$ -modules is the ζ -function lemma.

Lemma

or dep. on f and u .

For any section u of M $\exists P \in \mathbb{D}_X \ni s$, $\zeta \in \mathbb{C}[s]$, s.t

$$P(f^{s+1}u) = \zeta(s) f^s u \quad (\text{Sato, Bernstein})$$

Because $\mathbb{C}[s]$ is a P.I.D. $\exists$ smallest such $\zeta(s)$, and it is the ζ -function.

$$j_* \mathcal{M} = \mathbb{C}[f, f^{-1}] \mathcal{M}, \quad \mathcal{D} \mathcal{U} \subset \mathcal{M}$$

(23)

$\exists k \ll 0$, s.t. $f^k \mathcal{U}$ generates $j_* \mathcal{D} \mathcal{U}$

(choose $k < \text{any zero of } G(s)$)

$\Rightarrow \exists k \ll 0$, s.t. $f^k \mathcal{M}_0$ generates $j_* \mathcal{M}$

($\Rightarrow j_*(\text{hol})$ is still hol).

$\Rightarrow j_* \mathcal{M}$ is holonomic.

Now let's work over the punctured formal disk.

$$j_! f^s \mathcal{M}((s)) \xrightarrow{\sim} j_* f^s \mathcal{M}((s)) \quad \mathbb{C}((s))$$

by the b -function lemma

B -function lemma.

$j_* f^s \mathcal{M}((s))$ is generated by $j_* f^{s+k} \mathcal{M}_0((s))$.

for any k .

Lemma: $D_X(f^k M_0) = j_! * M \quad (= \text{Image } j_! M \rightarrow j_* M)$

for $k \gg 0$.

Pf: $j_* M / j_! * M$ is supported on F .

For $k \gg 0$, $f^k M_0 \subseteq j_! * M$

$M_0 \subseteq j_* M$ coherent s.t. $D_X M_0 = j_* M$. ▣

Lemma $\Rightarrow$

$$j_! f^s M((s)) \xrightarrow{\sim} j_* f^s M((s))$$

$$\begin{array}{ccc}
 U & \xhookrightarrow{j} & X \xhookrightarrow{i} F & f: X \rightarrow \mathbb{C} \\
 \parallel & & \parallel & \\
 f^*(\mathbb{C}^*) & & f^*(0) & \\
 M & & j_* M &
 \end{array}$$

Surjectivity $j_! \rightarrow j_*$, Injectivity is dual.

(25)

For the formal disc $\mathbb{C}[[s]]$, we have

$$j_! f^s \mathcal{M}[[s]] \hookrightarrow j_* f^s \mathcal{M}[[s]]$$

$$\text{and } j_* f^s \mathcal{M}[[s]] = \mathbb{D}_X[[s]] (f^{s+k} M_0), \quad k \gg 0.$$