Kari Vilonen

Mixed geometry

X space, scheme $\xrightarrow{IH^*}$ Vector space / abelian gp

mixed Geometry $\rightsquigarrow$ ultimately give a motive

"Basic theory" perverse sheaves, D-modules

Put some extra structure

What we get on a point is "the extra structure".

Q: What is the extra structure?

A: Instead of v. space, we put repr. of a group.

classical point of view, G = Galois group

X/F, F finite field, $G = \hat{\mathbb{Z}} = Gal(\bar{F}/F)$.

F local field, $\mathbb{Q}_p$ $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ ← structure of this group.

Work over $\mathbb{C}$:

The complex version of mixed Hodge structures.

V space V. two filtrations $W_\cdot$ $F^\cdot$ $\bar{F}^\cdot$

increasing decreasing

Induced filtrations $F^\cdot$ & $\bar{F}^\cdot$ are complimentary on $Gr_n^W V$

ie: $\underbrace{F^p \oplus \bar{F}^{n-p}}_{\text{induced filtrations}} = gr_n^W V$

A vector space

$$V = \bigoplus V^{p,q}, \quad N : V^{p,q} \to \bigoplus_{\substack{a<p \\ b<q}} V^{a,b} \quad \text{linear map.}$$

To go back: $W_n = \bigoplus_{p+q\leq n} V^{p,q}$

$$F^p = \exp(N)\Big(\bigoplus_{a\geq p} V^{a,b}\Big), \quad \bar{F}^q = \exp(-N)\Big(\bigoplus_{b\geq q} V^{a,b}\Big).$$

Second description:

$$MHS \iff \mathrm{Rep}\ V \rtimes \mathbb{G}_m \times \mathbb{G}_m$$

Solvable

$$\mathrm{Lie}(V) = \{ \mathfrak{v}_{a,b} \mid a, b \in \mathbb{Z} \}$$

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is solvable

Note: Cohom dim of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is 2.
Cohom. dim of MHS is 1.
} analogy

$\mathrm{Ind}_{\mathbb{G}_m \times \mathbb{G}_m}^{G}(\chi)$ is indecomposible pro-projective.

Claim:? Any subbundle of a p.proj is proj ?

$$H^i_{\mathcal{H}}(M) \overset{\text{def}}{=} \mathrm{Ext}^i_{MHS}(\mathbb{C}(0), M)$$

MHS

only non-zero in degrees 0 & 1.

$X \xrightarrow{f} pt$ smooth (proper)

$$\mathrm{RHom}_{MHM_X}(f^*\mathbb{C}(0), \mathcal{M}) = \mathrm{RHom}_{MHM}(\mathbb{C}(0), Rf_*M)$$

$$\uparrow$$

f smooth. $\Rightarrow$ $f^*\mathbb{C}(0)$ is of weight 0.

$\mathcal{H}^d(f^*\mathbb{C}(0)) = 0$, unless $d = \dim(X)$.

$\mathcal{O}_X \rightsquigarrow$ weight $d = \dim X$ $[-d]$.

Rmk: DR: $MHM_X \longrightarrow P(X)$

$\mathcal{M} \longmapsto \Omega_X^{\bullet} \otimes \Omega[\dim X]$

Comment.

Gufang explained: how to get a graded enhancement of $\mathcal{O}_0$ from the mixed story.

It's NOT straightforward!

Jantzen Conjecture

We consider category $\mathcal{O}$ or $\mathcal{O}'$.

We write τ for the - of the Chevally involution.

For an sl_2-triple. $\tau(x_\alpha) = y_\alpha$

$\tau(y_\alpha) = x_\alpha$

$\tau(h_\alpha) = h_\alpha$

$M = M_\lambda$ Verma module of h.w λ.

To construct $M^\vee$, we dualize the weight spaces of M & let $\mathfrak{g}$ act on $M^\vee$:

$$\text{by} \quad (uf)(v) = f(\tau(u)v).$$

Let v_+ be the h.w vector for M,

$v_+^\vee$ h.w vector for $M^\vee$, $(v_+\ v_+^\vee) = 1$.

We get a natural map $I: M \longrightarrow M^\vee$

$$v_+ \longmapsto v_+^\vee.$$

If we go to geometry. $j_! \xrightarrow{I} j_*$

The I gives us a contragredient pairing.

$$C: M \otimes M \longrightarrow \mathbb{C}$$

s.t. $\quad C(uv, v') = C(v, \tau(u) v')$ by

$$C(v, v') = I(v)(v').$$

<u>Note that:</u>

Call them M, $M^\vee$.

the underlying vector spaces of M_λ & $M_\lambda^\vee$ can be identified for various λ. as these modules are just abstractly $\cong U(n_-) v_f$

$$I_\lambda : M \longrightarrow M^\vee . \quad \lambda \in \mathfrak{h}^*.$$

This map can have zeros.

Intertwining Operators

<u>Rmk:</u> Some people like it to have poles.

KL: they fix λ. want to understand how the standard modules & irreducibles are related. at λ.

Jantzen: Look at the neighborhood at λ !!!

Similar to looking at $\mathbb{F}_p \leftrightarrow \mathbb{Q}_p \supseteq \mathbb{Z}_p$. $\odot$.

Let's consider two f.d vector spaces E & F.

$A = \mathbb{C}[[t]]$

$K = \mathbb{C}((t))$

$I_t : A \underset{\mathbb{C}}{\otimes} E \hookrightarrow A \underset{\mathbb{C}}{\otimes} F$ ← Free A-mod of rank n.

$\downarrow \underset{A}{\otimes} K$ s.t:

get an isom.

We have a basis of $A \underset{\mathbb{C}}{\otimes} F$ $[m_1, \dots, m_n]$

s.t: $\{ \underset{m_1'}{\underset{\|}{t^{d_1} m_1}}, \dots, \underset{m_n'}{\underset{\|}{t^{d_n} m_n}} \}$ is a basis of $A \underset{\mathbb{C}}{\otimes} E$. $d_1 | d_2 | \cdots | d_n$.

Can define descending filtrations F^ν, E^ν, s.t:

$$\left.\begin{aligned} A \underset{\mathbb{C}}{\otimes} E^\nu &= \langle m_i^\nu \mid d_i \geq \nu \} \\ A \underset{\mathbb{C}}{\otimes} F^\nu &= \langle m_i \mid d_i \leq -\nu \} \end{aligned}\right\} \text{ filtrations on } F \,\&\, E$$

$$E^\nu / E^{\nu+1} \cong F^{-\nu} / F^{-\nu+1} \quad \text{at } 0.$$

$$m \longmapsto t^{-m} I(m)$$

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Jantzen Conjecture

M Verma module of h. weight λ.

$M^\vee$ contragredient dual

$M \longrightarrow M^\vee \iff$ contragredient pairing on M.

$I(\lambda): M \to M^\vee$ $\quad$ (geom. $j_! \to j_*$)

Goal: We will define increasing filtr $W_\cdot' M$ $\quad$ $W_\cdot' M^\vee$ as follows.

We can define a universal Verma module / $U(\mathfrak{h}) = R$.

In the construction of a Verma module, $\otimes$ everything by $U(\mathfrak{h})$

How does $\mathfrak{h}$ act?

$$\begin{array}{ccc} U(\mathfrak{h}) \underset{\mathbb{C}}{\otimes} \mathfrak{h} & \longrightarrow & U(\mathfrak{h}) \\ 1 \otimes h & \longmapsto & h \end{array} \quad \leftarrow \text{h.w.}$$

M free mod / $U(\mathfrak{h})$ $\qquad M = U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R$

To define the Jantzen filtration, we restrict $\lambda + \xi t$ (line in $\mathfrak{h}^*$)
α any dominant wt

$I(0): E \to F \quad E, F$ fi dim $/\mathbb{C}$

$I(t): A \otimes_{\mathbb{C}} E \to A \otimes_{\mathbb{C}} F$ (two lattices), where $A = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$.

s.t $K \otimes_A (\quad) \xrightarrow{\text{isom}} K \otimes_A (\quad)$ get an isom.

$$E_r := \{ v \in A \otimes_{\mathbb{C}} E \mid t^r I(t) v \in A \otimes_{\mathbb{C}} F \}$$

The same defintion for F_r using $I(t)^{-1}: K \otimes_{\mathbb{C}} F \longrightarrow K \otimes_{\mathbb{C}} E$.

We get:

$\cdots E_{-2} \subseteq E_{-1} \subseteq E_0 = E$ $\qquad$ (evaluate at 0)
mod $/t$

$0 \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq F$.

Pf: $\exists$ basis $m_1, \cdots, m_n$ of $A \otimes_{\mathbb{C}} F$ s.t.

through the image of I

$m_i' = t^{d_i} m_i$ is a basis of $A \otimes_{\mathbb{C}} E$, $d_1 \mid d_2 \mid \cdots \mid d_n$.

$$E_r = \langle \{ m_k'(0) \mid d_k \geq -r \} \rangle$$

$$F_r = \langle \{ m_k(0) \mid d_k \leq r \} \rangle$$

$$E_{-r}/E_{-r-1} \longrightarrow F_r/F_{r-1}$$

$$v \longmapsto t^{-r} I(t) V(0).$$

$\square$

$M \longrightarrow M^\vee$ restriction $\eta + t\mathfrak{s}$ wt space and complete at 0.

$\Rightarrow$ Jantzen filtration on M_η & $M_\eta^\vee$.

The $W_i M$, $W_i M^\vee$ are $\mathcal{U}(\mathfrak{g})$- submodules.

(12)

<u>Fact</u>: $w_{-r}' M / w_{-r-1}' M \cong w_r' M^\vee / w_{r-1}' M^\vee$

Let's specialize to $\lambda = 0$. Can work in $\mathcal{O}_0$, $\mathcal{O}_0'$.

For $y, w \in W$, we have polys $P_{y,w}(q)$ defined by Gufang.

<u>Jantzen Conj</u>: $\lambda = 0$.

Ting's indexing. $w \in W \rightsquigarrow \overset{M_w}{\overset{\shortparallel}{M(w \cdot 0)}} \longleftrightarrow j_{w!} \mathbb{C}_{X_{ww_0}}[\ ]$

$X_e = pt$

<u>KL Conj</u>: $\boxed{M_y = \sum_{w \leq y} P_{wy}(1) L_w.}$

<u>Jantzen</u>: $\boxed{P_{w,y}(t) = \sum_i [Gr_{-i} M_y : L_w] t^i}$

$$gr_i^{W'} M \xleftarrow{\text{non-deg}} \text{contragredient pairing.}$$

Rmk: The filtration W' is the weight filtration. (not obvious)

Monodromy wt filtration: (BB)

Let Q be an object in any abelian category and $S \in End(Q)$ nilpotent

Then, there exists a unique filtration $\mu_\cdot$ on Q s.t.:

$$S : \mu_i Q \to \mu_{i-2} Q.$$

and

$$S^i \text{ induces an isom: } Gr_i^{\mu}(Q) \xrightarrow{\sim} Gr_{-i}^{\mu}(Q)$$

Rmk: $\mu_i Q = \sum_{k-j=i} \ker(S^{k+1}) \cap Im(S^j).$

$P_i = \ker(S : Gr_i Q \to Gr_{i-2} Q)$ primitive part.

$$Gr^{u}_{\bullet} Q = \bigoplus_{j \le 0} P_j \otimes \mathbb{Z}[S] / S^{-j+1}$$ graded mod

$\deg(P_i) = -i$

$\deg(S) = -2.$

Structure:

$$\begin{array}{ccc} & & P_{-2} \\ & P_{-1} & \\ P_0 & \downarrow S & P_{-2} S \\ & P_{-1} S & \\ & & P_{-2} S^2 \end{array}$$

$$0 \to \underset{\substack{\| \\ J_!}}{\ker(S)} \to Q \xrightarrow{S} Q \to \underset{\substack{\| \\ J_*}}{\operatorname{coker}(S)} \to 0$$

Filtration on Q induces a filtration on $\underset{\substack{\| \\ J_!}}{\ker(S)}$ & $\underset{\substack{\| \\ J_*}}{\operatorname{Coker}(S)}$.

$$\begin{cases} W'_i J_! = J_! \cap \operatorname{Im}(s^{-i}) & i \leq 0 \quad \text{Stabilizer} \leftarrow \text{Negative degrees} \\ W'_i J_* = (\ker s^{i} + \operatorname{Im} s)/\operatorname{Im} s & \leftarrow \text{Filtration in pos. degrees} \end{cases}$$

(This is the weight filtration).

Strict exact sequences

$$(*) \quad 0 \to (J_!, W'_\cdot) \to (Q, \mu) \to (\bar{Q}, \mu_{\cdot -1}) \to 0$$

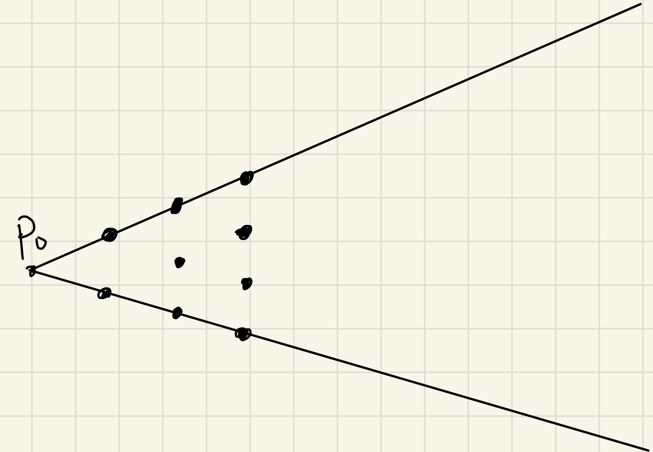
$$0 \to (\bar{Q}, \mu_{\cdot +1}) \to (Q, \mu) \to (J_*, W'_\cdot) \to 0$$

where $\bar{Q} := Q/\ker(s) \circlearrowleft \bar{s}$

<u>Note</u>: $\bar{Q} \cong \operatorname{Im}(s)$.

$$\mathrm{Gr}\,\mathcal{J}_! = \bigoplus_{j\le 0} P_j\, s^{-j}$$

$$\mathrm{Gr}\,\mathcal{J}_* = \bigoplus_{j\le 0} P_j .$$



Apply Gr to (*):

$$0 \to \underset{\mathrm{Gr}\,\mathcal{J}_!}{\underset{\parallel}{\bigoplus_{j\le 0} P_j\, s^{-j}}} \longrightarrow \bigoplus_{j\le 0} P_j \otimes \mathbb{Z}[s]/s^{-j+1} \longrightarrow \bigoplus_{j\le 0} P_j\, \mathbb{Z}[s]/s^{-j} \to 0.$$

$$\boxed{Q = \Xi_f}$$

$f : X \to \mathbb{C}$

$$\underset{f^{-1}(\mathbb{C}^*)}{\underset{\parallel}{U}} \hookrightarrow X \overset{i}{\hookleftarrow} \underset{\text{closed.}}{F}$$

$$P(U) \xrightarrow{\text{pass}} P(X)$$

Nearby cycle construction

$$\begin{tikzcd} & j_{!*} \arrow[dr, hook] & \\ j_! \arrow[ur, two heads] \arrow[dr, hook] & & j_* \\ & (?) \arrow[ur, two heads] & \end{tikzcd}$$

$$\Xi_f$$

when ! & * collide.

see: Ξ_f

$$\begin{array}{ccccc} U & & X & & F \\ \parallel & & \parallel & & \parallel \\ X_w & \hookrightarrow & \overline{X_w} & \hookleftarrow & \partial \overline{X_w} \end{array}$$

← a principal divisor

$S =$ log. unipotent monodromy.

Inputs:

KL, mixed geometry

Koszul duality

(statement)

$\Rightarrow$

more things

12 Nov 2020:

Elementary to define the Jantzen filtration on $M_\mathcal{R}$ & $M_\mathcal{R}^\vee$.

Next step is to show it is the weight filtration (shifted)

We use mixed geometry. Gufang chose to use Hodge modules.

We have the category $MHM(X)$

Recall the objects are $(\mathcal{M}, F_\bullet\mathcal{M}, W_\bullet\mathcal{M})$

D_X-mod D_X-filtration by D_X-modules.

- $D_X(n)\, F_m\mathcal{M} \subseteq F_{m+n}\mathcal{M}$

- $D_X\, W_m\mathcal{M} \subseteq W_m\mathcal{M}$

Recall: $\mathcal{M}$ hol regular.

The quotient $W_r\mathcal{M}/W_{r-1}\mathcal{M}$ is of weight r. It is semi-simple.

If $Gr_r^W \mathcal{M} \neq 0$, we say that $\mathcal{M}$ is of weight r

$\mathcal{M}$ has wt $\geq r$ $\iff$ $Gr_i^W\mathcal{M} = 0$ for $i < r$

M has wt $\leq r \iff Gr_i^W M = 0$ for $i > r$.

If M is a complex, then M has wt $\leq w \iff H^k(M)$ is of wt $\leq k+w$.

$\geq w \iff$ —— " —— $\geq k+w$

We have 6 functor formalism

f_* $f^!$ increase wts

$f_!$ f^* decrease wts.

Socle filtration: of M. ← any artinian cat

$S_0 M =$ maximal semi-simple sub obj.

$S_i M =$ pullback of $S_0(M/S_{i-1}M)$.

Assume M is a mixed Hodge module, M^{red} forget the mixed Hodge structure

They both have their socle filtrations.

These socle filtrations are different in general as one can see already when X is a point.

<u>Def:</u> We say that M is partially pure, $*$-pure or $!$-pure of weight w, if $i_x^* M$ is pure of wt w for all $x \in X$, $i_x : \{x\} \hookrightarrow X$.

if $i_x^! M$ ——— " ———

Very strong condition.

It's true for pure objects on flag manifolds w.r.t. N- or K- orbits.

<u>Lemma</u> (Beilinson–Bernstein)

$Y \overset{i}{\subseteq} X$ locally closed, M pure of wt w on Y. Then for any $N \subseteq \mathcal{H}^0(i_* M)$ in MHM s.t its subquotients are all $!$-pointwise pure.

We have : $S_i N = W_{w+i} N$.

Furthermore, the mixed and non-mixed socle filtrations coincide.

Nearby cycle Construction:

Consider X, $f : X \to \mathbb{C}$ smooth

$$\begin{array}{ccccc} U & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookleftarrow} & F \\ \| & & & & \| \\ f^{-1}(\mathbb{C}^*) & & & & f^{-1}(0) \end{array}$$

Consider a holonomic module M on U. We will analyze $j_* M$ & $j_! M$.

We will work locally near a point in F.

Thus, we can assume that U is stein. (affine in alg cat)

Hence, M is generated by a $\mathcal{O}_U$ - Coherent submodule M_0.

A serious input into D-modules is the b-function lemma.

Lemma

For any section u of M $\exists$ $P \in D_X[s]$, $b \in \mathbb{C}[s]$, s.t ← dep. on f. and u.

$$\boxed{P(f^{s+1} u) = b(s) f^s u}$$ (Sato, Bernstein)

Because $\mathbb{C}[s]$ is a P.I.D. $\exists$ smallest such $b(s)$, and it is the b-function.

$j_* \mathcal{M} = \mathbb{C}[f, f^{-1}]\, \mathcal{M}$. $\qquad \mathcal{D}\mathcal{U} \subset \mathcal{M}$

$\exists\ k \ll 0$, s.t: $f^k \mathcal{U}$ generates $j_* \mathcal{D}\mathcal{U}$

Choose k < any zero of $b(s)$

$\Rightarrow \exists\ k \ll 0$, s.t $f^k \mathcal{M}_0$ generates $j_* \mathcal{M}$

($\Rightarrow j_*(\text{hol})$ is still hol).

$\Rightarrow$ $j_* \mathcal{M}$ is holonomic.

Now let's work over the punctured formal disk.

$$\boxed{j_! f^s \mathcal{M}((s)) \xrightarrow{\sim} j_* f^s \mathcal{M}((s))} \qquad \mathbb{C}((s)).$$

by the b-function lemma

<u>B-function lemma.</u>

$j_* f^s \mathcal{M}((s))$ is generated by $j_* f^{s+k} \mathcal{M}_0((s))$.

for any k.

__Lemma:__ $D_X(f^k \mathcal{M}_0) = j_{!*}\mathcal{M}$ $\quad (= \mathrm{Image}\ j_!\mathcal{M} \to j_*\mathcal{M})$

for $k \gg 0$.

Pf: $j_*\mathcal{M}/j_{!*}\mathcal{M}$ is supported on F.

For $k \gg 0$, $f^k \mathcal{M}_0 \subseteq j_{!*}\mathcal{M}$

$\mathcal{M}_0 \subseteq j_*\mathcal{M}$ coherent s.t $D_X \mathcal{M}_0 = j_*\mathcal{M}$. ▨

Lemma $\Rightarrow$ $$\boxed{j_! f^s \mathcal{M}((s)) \xrightarrow{\sim} j_* f^s \mathcal{M}((s))}$$

$$\begin{array}{ccccc} U & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookleftarrow} & F \\ \| & & & & \| \\ f^{-1}(\mathbb{C}^*) & & & & f^{-1}(0) \end{array} \qquad f: X \to \mathbb{C}$$

$\mathcal{M}$ $\qquad j_*\mathcal{M}$

Surjectivity $j_! \to j_*$, Injectivity is dual.

For the formal disc $\mathbb{C}[[s]]$, we have

$$j_! f^s \mathcal{M}[[s]] \hookrightarrow j_* f^s \mathcal{M}[[s]]$$

and $j_* f^s \mathcal{M}[[s]] = \mathcal{D}_X[[s]](f^{s-k} \mathcal{M}_0)$, $k \gg 0$.

Lemma: Assume that $\mathcal{M}$ is a mixed Hodge module such that: all its subquotients are !-pointwise pure.

Then:
$$(S_i \mathcal{M})^{red} = S_i(\mathcal{M}^{red}).$$

proof:

Clearly, $(S_0\mathcal{M})^{red} \subseteq S_0 \mathcal{M}^{red}$. Show the opposite.

$$\begin{tikzcd} & Z \arrow[r, hook, "j"] \arrow[dr, "i"'] & \bar{Z} \arrow[d, "\bar{i}"] \\ \arrow[ur, "\text{smooth}"] & & X \end{tikzcd}$$

Irreducible local system $\mathcal{L}$, i.e: V. B. with a flat connection on Z.

$$\bar{i}_* j_{!*}\mathcal{L} \hookrightarrow \mathcal{M}^{red}.$$

$\Updownarrow$ by adjunction.

$$j_{!*}\mathcal{L} \longrightarrow \mathcal{H}^0(\bar{i}^! \mathcal{M})^{red}.$$

We shrink Z, s.t: $i^! \mathcal{M}$ is smooth on Z.

$$\begin{tikzcd} j_{!*}\mathcal{L} \arrow[r] & \mathcal{H}(i^! M)^{\mathrm{red}} \\ j_!\mathcal{L} \arrow[u, two heads] \arrow[ur] & \end{tikzcd} \quad \xrightarrow{j^*} \quad \begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, equal] & \mathcal{H}^0(i^! M)^{\mathrm{red}} \\ \mathcal{L} \arrow[ur] & \text{pure} \arrow[u] \end{tikzcd}$$

$\mathcal{L}$ is a summand & thus has compatible Hodge structure.

Reverse the argument. $\square$

<u>Lemma</u> (Beilinson–Bernstein)

$Y \overset{i}{\subseteq} X$ smooth locally closed, M pure Hodge mod of wt w on Y.

$N \subseteq \mathcal{H}^0(i_* M)$ a subMHM s.t all subquotients are all !-pointwise pure.

Then: $\boxed{S_i N = W_{w+i} N.}$

$\overset{\text{Lemma}}{\Longrightarrow}$ $\quad S_i N^{\mathrm{red}} = (W_{w+i} N)^{\mathrm{red}}$.

<u>Proof:</u> Prove it by induction on i & $\dim(\operatorname{supp} N)$.

Note the wt $\mathcal{H}^i(i_* M) \geq w$ because i_* increases weights.

$$S_{-1} N = W_{w-1} N = 0$$

$$S_0 (\mathcal{H}^0(i_* M)) = i_{!*} M = W_w (\mathcal{H}^0(i_* M))$$

$$\Rightarrow S_0 N = W_w N$$

Of course $W_{w+i} \subseteq S_i N$ because $gr^W_\cdot N$ is semi-simple.

Have to show the opposite.

Assume $W_{w+i} N \subsetneq S_i N$

$\Rightarrow \exists$ an irreducible summand A of wt $> w+i$ of $S_i N / \underbrace{W_{i+w-1} N}_{\substack{\parallel \\ S_{i-1} N}}$

Let's assume the support of A is a point

If not, just cut to support generically by a normal slice.

$i : \{x\} \hookrightarrow X \qquad A = i_* \mathcal{L}, \qquad \mathcal{L}$ pure of wt $>$ wt $-i$

$$\begin{tikzcd}
0 \arrow[r] & W_{wt-i-1}N / W_{wt-i-2}N \arrow[r] & S_i N / W_{wt-i-2}N \arrow[r] & S_i N / W_{wt-i-1}N \arrow[r] & 0 \\
0 \arrow[r] & W_{wt-i-1}N / W_{wt-i-2}N \arrow[r] & B \arrow[r] & A \arrow[r] \arrow[u, hook, "\oplus"] & 0
\end{tikzcd}$$

Bottom exact sequence is a non-trivial extension even after reduction.

(By previous lemma $\quad S_i N^{red} = (S_i N)^{red}$).

Have a non-trivial class in

$$\mathrm{Ext}^1_{MHM}(A, W_{wt-i-1}N / W_{wt-i-2}N) = \mathrm{Ext}^1_{MHM}(\mathcal{L}, \overbrace{i^!(W_{wt-i-1}N / W_{wt-i-2}N)}^{M \,=:})$$

$$\downarrow$$

$$\mathrm{Ext}^1(A, \text{——} \text{''} \text{——}) = \mathrm{Ext}^1(\mathcal{L}, \text{——} \text{''} \text{——})$$

$M := i^!(W_{wt-i-1}N / W_{wt-i-2}N)$ lies in deg ≥ 0 and is pure of wt $wt-i-1$.

wt > wt i $\qquad\qquad$ wt wt i

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{MH}(L, H^0(M)) \arrow[r] & \mathrm{Ext}^1_{MH}(L, M) \arrow[r] \arrow[d] & \mathrm{Hom}_{MH}(L, \underbrace{H^1(M)}_{=0}) \arrow[r] \arrow[d] & 0 \\
\text{reduced.} \quad 0 \arrow[r] & & \mathrm{Ext}^1_{MH}(L, M) \arrow[r, phantom, "="] & \mathrm{Hom}(L, H^1 M) \arrow[r] & 0.
\end{tikzcd}
$$

For Verma's the cosocle filtrations coincides with the wt filtration up to a shift.

The dual Verma's the socle ——— " ———