References:
Zagier, Applications of the representation theory of finite groups 2004
Serre, Linear Representations of Finite Groups 1977

More on writing a representation as a direct sum of irreducible ones

Let G be a finite group.
Let $\{(V_i, \pi_i)\}_{i\in I}$ be a full set of non-isomorphic irreducible representations of G.

Recall,

- Any representation V can be written as
$$V \simeq \bigoplus_{i\in I} \underbrace{V_i \oplus \cdots \oplus V_i}_{k_i \text{ copies}}$$
- $\mathrm{Hom}_{\mathbb{C}}(A, V) \simeq V \otimes_{\mathbb{C}} A \simeq \underbrace{V \oplus \cdots \oplus V}_{\dim A \text{ copies}}$

So $V \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} A_i \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(B_i, V_i)$ for some k_i-dimensional $\mathbb{C}$-vector spaces A_i and B_i.

Lemma If $A_i = \mathrm{Hom}_G(V_i, V)$, $B_i = \mathrm{Hom}_G(V, V_i)$
then $V \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} A_i \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(B_i, V_i)$

ie. Let V be a representation of G.
Then we have canonical G-equivariant isomorphisms
$$\bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) \xrightarrow{\sim} V \qquad \text{and} \qquad V \xrightarrow{\sim} \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V, V_i), V_i)$$
$$x \otimes \phi \longmapsto \phi(x) \qquad\qquad v \longmapsto (\underset{\text{homomorphism}}{\phi} \longmapsto \phi(v))$$

Conversely, if $V \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} A_i \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(B_i, V_i)$ then
$A_i = \mathrm{Hom}_G(V_i, V)$, $B_i = \mathrm{Hom}_G(V, V_i)$.

Proof

$\boxed{\text{To show: } V \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V, V_i), V_i)}$

$$\Leftrightarrow \bigoplus_{j\in I} \underline{V_j} \oplus \cdots \oplus V_j \simeq \bigoplus_{j\in I} \left(\underline{\bigoplus_{i\in I} V_i \otimes \mathrm{Hom}_G(V_i, V_j)}\right) \oplus \cdots \oplus \left(\quad \right) \simeq \bigoplus_{j\in I} \left(\underline{\mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V_j, V_i), V_i)}\right) \oplus \cdots \oplus \left(\quad \right)$$

Since both isomorphisms are additive,

$$\Leftarrow V_j \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} \mathrm{Hom}_G(V_i, V_j) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V_j, V_i), V_i)$$

By Schur's lemma, $\mathrm{Hom}_G(V_i, V_j) \simeq \mathbb{C}$ if $i = j$,
$\mathrm{Hom}_G(V_i, V_j) = \{0\}$ if $i \neq j$.

$$\Leftrightarrow V_j \simeq (V_j \otimes \mathbb{C}) \oplus \left(\bigoplus_{\substack{i\in I \\ \text{and} \\ i\neq j}} V_i \otimes_{\mathbb{C}} \{0\}\right) \simeq (\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V_j)) \oplus \left(\bigoplus_{\substack{i\in I \\ \text{and} \\ i\neq j}} \mathrm{Hom}_{\mathbb{C}}(\{0\}, V_i)\right)$$

$$\Leftrightarrow \boxed{V_j \simeq V_j \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V_j)}$$

Recall, $V \otimes_{\mathbb{C}} A \simeq \mathrm{Hom}_{\mathbb{C}}(A, V) \simeq \underbrace{V \oplus \cdots \oplus V}_{\dim A \text{ copies}}$

So $V \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V_j) \simeq V_j$.

So now we have shown that:
$$V \simeq \bigoplus_{i\in I} V_i \otimes_{\mathbb{C}} \mathrm{Hom}_G(V_i, V) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V, V_i), V_i)$$

Group algebra

$\mathbb{Q}[x] = \{a_0 x^0 + a_1 x^1 + \cdots, \text{ where } a_i \in \mathbb{Q}\}$ $\quad x^i$

Group algebra $\mathbb{C}[G]$ is the set of linear combinations $\sum_{g\in G} \alpha_g [g]$ $(\alpha_g \in \mathbb{C})$ formal symbol

$\mathbb{C}[G]$ is a representation of G, with G-action $g_1 [g] = [g_1 g]$.

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<u>Theorem</u> $\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}_{\mathbb{C}}(V_i)$ $\leftarrow$ $\mathrm{End}_{\mathbb{C}}(V_i)$
= endomorphism algebra of V_i
= set of all homomorphisms $V_i \to V_i$
= $\mathrm{Hom}_{\mathbb{C}}(V_i, V_i)$

<u>Proof</u>

Recall the previous result, $V \simeq \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(V, V_i), V_i)$

Apply this to $V = \mathbb{C}[G]$,

$$\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(\mathbb{C}[G], V_i), V_i)$$

To show: $\boxed{\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}_{\mathbb{C}}(V_i)}$

$\Leftrightarrow \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}_G(\mathbb{C}[G], V_i), V_i) \simeq \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(V_i, V_i)$

$\Leftarrow \mathrm{Hom}_G(\mathbb{C}[G], V_i) \simeq V_i$

$\boxed{\text{STP: } \mathrm{Hom}_G(\mathbb{C}[G], V) \simeq V \text{ for any representation } V \text{ of } G.}$

Let V be any representation of G.

Define a linear map $\Phi : \mathrm{Hom}_G(\mathbb{C}[G], V) \to V$, $A \mapsto A(1)$

- $A : \mathbb{C}[G] \to V$, $[1] \mapsto A(1)$, $[g] \mapsto g A(1)$
- A is G-equivariant
- $(g \cdot A)(h) = A(hg)$

$\boxed{\text{To show: } \Phi \text{ is i) bijective, ii) } G\text{-equivariant}}$

i) Φ has an inverse $\Psi : V \longrightarrow \mathrm{Hom}_G(\mathbb{C}[G], V)$, $v \longmapsto A_v$ such that $A_v(1) = v$

$\left[A_v : \mathbb{C}[G] \to V, \ [1] \mapsto v, \ [g] \mapsto g.v \right]$

Since $\Phi(\Psi(v)) = \Phi(A_v) = A_v(1) = v \Rightarrow \Phi \circ \Psi = id$

$\Psi(\Phi(A)) = \Psi(A(1)) = A \Rightarrow \Psi \circ \Phi = id$,

so Φ is an isomorphism.

ii) $\boxed{\text{To show: } \Phi(g \cdot A) = g(\Phi(A))}$

$\Phi(g \cdot A) = (g \cdot A)(1) = A(1 \cdot g) = A(g)$ (definition of G-action on A)

$g(\Phi(A)) = g(A(1)) = A(g \cdot 1) = A(g)$ (A is G-equivariant)

So $\Phi(g \cdot A) = g(\Phi(A))$

So Φ is G-equivariant.

So $\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}(V_i)$.

We have now shown that

$$\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}_{\mathbb{C}}(V_i)$$

Let C be the set of conjugacy classes in G and
R be the set of isomorphism classes of irreducible representations of G.
[Note: $I \simeq R$]

$R = \{ \pi_1 \simeq \pi_2 \simeq \pi_3, \ \pi_4 \simeq \pi_5, \ldots \}$

Let C be the set of conjugacy classes in G and
R be the set of isomorphism classes of irreducible representations of G.
[Note : $I \simeq R$]

$$R = \{ \pi_1 \simeq \pi_2 \simeq \pi_3 , \pi_4 \simeq \pi_5 , \ldots \}$$

We can write character as $\chi_\pi(e)$ for $\pi \in R$ and $e \in C$.

<u>Corollary</u> $\sum_{\pi \in R} (\dim \pi)^2 = |G|$

<u>Proof</u> Consider the dimension on both sides of
$\mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}(V_i)$

$$
\begin{aligned}
|G| = \dim(\mathbb{C}[G]) &= \dim\left(\bigoplus_{i \in I} \mathrm{End}(V_i)\right) \\
&= \sum_{i \in I} \dim(\mathrm{End}(V_i)) \\
&= \sum_{i \in I} [\dim(V_i)]^2 \\
&= \sum_{i \in I} (\dim \pi)^2
\end{aligned}
$$

$\dim V_i$

$\dim V_i$

For example, $S_3 = \{ Id, (12), (23), (13), (123), (132) \}$

- $V = \mathbb{C}$, $1 = g \mapsto 1$ $\dim = 1$
- $V = \mathbb{C}$, $\varepsilon_3 : g \mapsto \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$ $\dim = 1$
- $V = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \}$, $St_3 : g \mapsto$ permutation $\dim = 2$

$$\sum_{i \in I} (\dim \pi_i)^2 = 1^2 + 1^2 + 2^2 = 6 = |S_3|$$

<u>Corollary</u> $|R| = |C|$

<u>Proof</u>
Consider the centre $Z(\mathbb{C}[G]) = \{ z \in \mathbb{C}[G] \mid hzh^{-1} \in \mathbb{C}[G] \text{ for all } h \in \mathbb{C}[G] \}$,
a basis for $Z(\mathbb{C}[G])$ is

$$\{ e_c = \sum_{g \in c} [g] \mid c \in C \}$$

since
$$
\begin{aligned}
[g_1] \left(\sum_{g \in c} [g] \right) [g_1^{-1}] &= \sum_{g \in c} [g_1 g g_1^{-1}] \\
&= \sum_{g_1 g g_1^{-1} \in c} [g_1 g g_1^{-1}] \\
&= \sum_{g \in c} [g]
\end{aligned}
$$

Consider the centre $Z(\mathrm{End}_{\mathbb{C}}(V_i))$,
$\mathrm{End}_{\mathbb{C}}(V_i)$ is the matrix algebra $M_{\dim V_i}(\mathbb{C})$.
It has 1-dimensional centre

$\leftarrow \dim V_i \rightarrow$ $\dim V_i$

$$\left\{ \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix} \;\middle|\; a \in \mathbb{C} \right\}$$

since $\phi \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \phi^{-1} = a \phi \phi^{-1} = aI$

So
$$
\begin{aligned}
|C| &= \left| \{ e_c = \sum_{g \in c} [g] \mid c \in C \} \right| \\
&= \dim Z(\mathbb{C}[G]) \\
&= \dim Z\left(\bigoplus_{i \in I} \mathrm{End}_{\mathbb{C}}(V_i) \right) \quad \text{since } \mathbb{C}[G] \simeq \bigoplus_{i \in I} \mathrm{End}(V_i) \\
&= \sum_{i \in I} \dim Z(\mathrm{End}_{\mathbb{C}}(V_i)) \\
&= \sum_{i \in I} 1 \\
&= |I| \\
&= |R| \quad \text{since } I \simeq R
\end{aligned}
$$

For example, in S_3,

$$C = \{ e_{Id}, e_{12}, e_{123} \} \qquad R = \{ 1, \varepsilon_3, St_3 \}$$

$$|C| = 3 = |R|$$

Corollary (Second orthogonality relation)

Let $c_1, c_2 \in C$. Then

$$\sum_{\pi \in R} \chi_\pi(c_1)\overline{\chi_\pi(c_2)} = \begin{cases} \frac{|G|}{|c_1|} & \text{if } c_1 = c_2 \\ 0 & \text{otherwise} \end{cases}$$

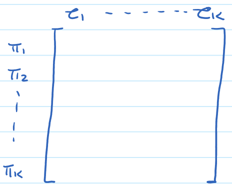
Proof

Recall,

- $|C| = |R|$
- $\frac{1}{|G|}\sum_{g \in G} \chi_\pi(g)\overline{\chi_{\pi'}(g)} = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \text{otherwise} \end{cases}$

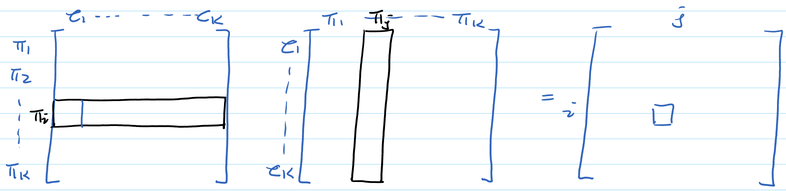
ie. $\frac{1}{|G|}\sum_{c \in C} |c|\,\chi_\pi(c)\overline{\chi_{\pi'}(c)} = \delta_{\pi,\pi'}$

Consider a matrix A with $A_{ij} = \frac{|c_j|^{\frac{1}{2}}\chi_{\pi_i}(c_j)}{|G|^{\frac{1}{2}}}$,



- A is square since $|R| = |C|$
- A is unitary since:

$A \times \bar{A}^t = B$

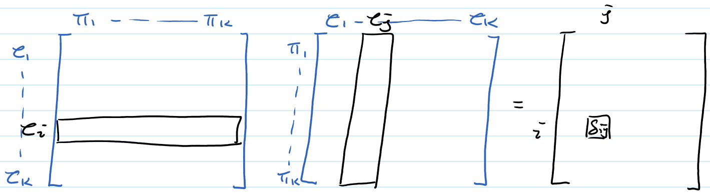


$$B_{ij} = \sum_{c \in C} \frac{|c|^{\frac{1}{2}}\chi_{\pi_i}(c)}{|G|^{\frac{1}{2}}} \, \frac{|c|^{\frac{1}{2}}\overline{\chi_{\pi_j}(c)}}{|G|^{\frac{1}{2}}}$$

$$= \frac{1}{|G|}\sum_{c \in C} |c|\,\chi_{\pi_i}(c)\overline{\chi_{\pi_j}(c)} = \delta_{ij}$$

So $A\bar{A}^t = I$.

So $\bar{A}^t A = I$



$$\frac{1}{|G|}\sum_{\pi \in R} |c_i|^{\frac{1}{2}}|c_j|^{\frac{1}{2}}\chi_\pi(c_i)\overline{\chi_\pi(c_j)} = \delta_{i,j}$$

So $\sum_{\pi \in R} \chi_\pi(c_i)\overline{\chi_\pi(c_j)} = \begin{cases} \frac{|G|}{|c_i|} & \text{if } c_i = c_j \\ 0 & \text{otherwise} \end{cases}$

More on characters

Character of a representation (V, π) is the function $\chi_\pi : G \to \mathbb{C}$

$$\chi_\pi(g) = tr(\pi(g), V)$$

Define $(\chi_{\pi_1} | \chi_{\pi_2}) = \frac{1}{|G|}\sum_{g \in G} \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}$. This is a scalar product.

The First Orthogonality relation can now be written as:

Let (V, π) and (V', π') be two irreducible representations of G.

$$(\chi_\pi | \chi_{\pi'}) = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \text{if } \pi \not\simeq \pi' \end{cases}$$

[The irreducible characters form an orthonormal system.]

Let (V,π) be a representation of G, with character χ_π.
Let $V = V_1 \oplus \cdots \oplus V_k$ where V_i's are irreducible representations, with characters χ_{π_i}.

<u>Theorem</u> If W is an irreducible representation with character ϕ, then the number of V_i's isomorphic to W is $(\chi_\pi \mid \phi)$.

<u>Proof</u>

Using, the character of $V_1 \oplus V_2$ is $\chi_{\pi_1} + \chi_{\pi_2}$

$$(\chi_\pi \mid \phi) = (\chi_{\pi_1} + \cdots + \chi_{\pi_k} \mid \phi)$$
$$= (\chi_{\pi_1} \mid \phi) + \cdots + (\chi_{\pi_k} \mid \phi) \quad (\text{scalar product})$$

Recall, $(\chi_{\pi_i} \mid \chi_{\pi_j}) = \begin{cases} 1 & \text{if } \pi_i \simeq \pi_j \\ 0 & \text{otherwise} \end{cases}$

So # V_i isomorphic to W $= (\chi_\pi \mid \phi)$

Since $(\chi_\pi \mid \phi)$ does not depend on the decomposition $V = V_1 \oplus \cdots \oplus V_k$, # V_i isomorphic to W does not depend on the decomposition.

If two representations have the same character χ, then given any irreducible representation W with character ϕ, $(\chi \mid \phi)$ = # times W occurs is the same for both representations.

So the two representations with the same character are isomorphic.

a way to check if representations are isomorphic

<u>Theorem</u>
Let (V,π) be a representation of G with character χ_π. Then
$(\chi_\pi \mid \chi_\pi) = 1$ if and only if V is irreducible.

a way to check if a representation is irreducible

<u>EG</u> For $S_3 = \{ Id, (12), (23), (13), (132), (123) \}$:

$V = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \}$
$\pi : g \mapsto$ permutation

character table

e	Id $\{Id\}$	(12) $\{(12),(23),(13)\}$	(123) $\{(123),(132)\}$
$\chi_\pi(g)$	2	0	-1

$$(\chi_\pi \mid \chi_\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g) \overline{\chi_\pi(g)}$$
$$= \frac{1}{6} \left(2^2 + 3(0)^2 + 2(-1)^2 \right)$$
$$= 1$$

So V is an irreducible representation