Representation of finite groups (2)

Reference: Zagier, Applications of the representation theory of finite groups 2004

Last week

- representation (V, π) of group G:
  $\mathbb{C}$-vector space V
  $\pi: G \longrightarrow GL(V)$
  $g \longmapsto \pi(g)$ ← bijective linear transformation
  $\pi(g): V \to V$
  $v \mapsto \pi(g)v$

- $\dim \pi = \dim(V)$

- V and V' are isomorphic ($V \simeq V'$) if there exists a G-equivariant isomorphism from V to V'
  ↳ $A: V \to V'$ s.t. $A(\pi(g)v) = \pi'(g)A(v)$

* If V is a representation of G then,
  - $V \otimes_{\mathbb{C}} A \simeq \mathrm{Hom}_{\mathbb{C}}(A, V) \simeq V \oplus \cdots \oplus V$ are representations

    $\mathrm{Hom}_{\mathbb{C}}(A, V)$: set of homomorphisms from A to V

    $V \otimes_{\mathbb{C}} A$: quotient of $\{v \otimes w \mid v \in V, w \in W\}$ by $c(v, w) = (cv, w) = (v, cw)$ for $c \in \mathbb{C}$

    ie with basis $(V \otimes_{\mathbb{C}} A) = \text{basis}(V) \times \text{basis}(A)$

  - $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a representation
  - $\mathrm{Hom}_{\mathbb{C}}(V, A) \simeq V^* \oplus \cdots \oplus V^*$ is also a representation

- $V \otimes_G V$ as the quotient of $V \otimes_{\mathbb{C}} V'$ by $gv \otimes v' = v \otimes gv'$
  $\mathrm{Hom}_G(V, V')$ = set of G-equivariant linear transformations from V to V'.
  "commutative with g"
  [NOT representations]

- V is irreducible if it contains no proper sub-space which is invariant under the action of G.

* Any representation of G is a direct sum of irreducible ones.

(irreducible)
what does $\mathrm{Hom}_G(V, V')$ look like?
$V \simeq V'$ then $\mathbb{C}$
$V \not\simeq V'$ then $\{0\}$

Schur's Lemma

Let V and V' be two irreducible representations of G. ①
Then the $\mathbb{C}$-vector space $\mathrm{Hom}_G(V, V')$ is 0-dimensional if $V \not\simeq V'$ and 1-dimensional if $V \simeq V'$. ②
The space $\mathrm{Hom}_G(V, V)$ is canonically isomorphic to $\mathbb{C}$.

↳ ∃ a most natural isomorphism

Proof

① Assume $V \not\simeq V'$
To show: $\mathrm{Hom}_G(V, V') = \{0\}$

Let $\phi \in \mathrm{Hom}_G(V, V')$ $\quad$ ϕ is G-equivariant
Then, to show: $\phi = 0$.

To show: $Im(\phi) = 0$.

Let $w \in Im(\phi)$
Then $w = \phi(u)$ for some $u \in U$.

$$gw = g\phi(u) = \phi(gu) \in Im(\phi)$$

$\therefore$ $Im(\phi) \subseteq U'$ is G-invariant

Since U' is irreducible, $Im(\phi)$ is trivial

$\therefore$ $Im(\phi) = \{0\}$ or $Im(\phi) = V'$

Let $v \in Ker(\phi)$.
Then $\phi(gv) = g \cdot \phi(v)$ since ϕ is G-equivariant
$= g \cdot 0$
$= 0$
So $gv \in Ker(\phi)$
So $Ker(\phi)$ is G-invariant
Since V is irreducible,
$Ker(\phi) \leq V$ is a trivial subspace.
So $Ker(\phi) = \{0\}$ or $Ker(\phi) = V$

If $\underbrace{Im(\phi) = U'}_{\to \text{ surjective}}$, $\underbrace{Ker(\phi) = \{0\}}_{\to \text{ injective}}$,
so ϕ is bijective
But $V \not\cong V'$
This is a contradiction.

$\therefore$ $Im(\phi) = 0$
$\therefore$ $\phi = 0$
$\therefore$ $Hom_G(V, U') = 0$

2 Assume $V \simeq U'$
Let $V = U'$.

$\boxed{\text{To show: } Hom_G(U,U) \simeq \mathbb{C}}$

Define a linear transformation
$\psi: Hom_G(U,U) \to \mathbb{C}$
$\phi \mapsto \phi$'s eigenvalue

To show: ψ is
- well-defined
- bijective

Let $\phi \in Hom_G(U,U)$.
Let λ be an eigenvalue of ϕ. (λ exists because $\mathbb{C}$)

Let $v \in Ker(\phi - \lambda)$

$(\phi-\lambda)(gv) = \phi(gv) - \lambda(gv)$
$= g\phi(v) - g(\lambda v)$ since ϕ is G-equivariant
$= g(\phi(v) - \lambda v)$ λ is scalar
$= g0$ since λ is an eigenvector of ϕ

$= 0$

So $gv \in \ker(\phi - \lambda)$
So $\ker(\phi-\lambda) \subseteq V$ is G-invariant

Also $\ker(\phi-\lambda) \neq \{0\}$ since eigenspace $\neq \{0\}$
So $\ker(\phi-\lambda) = V$
So $\phi - \lambda = 0$
So $\phi = \lambda$

So each ϕ only has 1 eigenvalue;
if ϕ_1, ϕ_2 have the same eigenvalue λ, then $\phi_1 = \lambda = \phi_2$;
each λ corresponds to 1 $\phi = \lambda I$

So the linear map $\phi : \mathrm{Hom}_G(V,V) \to \mathbb{C}$ is bijective.
$$\phi \mapsto \lambda$$

So $\mathrm{Hom}_G(V,V) \simeq \mathbb{C}$

and $\mathrm{Hom}_G(V,V') \simeq \mathrm{Hom}_G(V,V)$ is 1-dimensional.

Character of a representation

Let (V,π) be an irreducible representation of G.

The <u>character</u> of (V,π) is the function

$$\chi_\pi(g) = \mathrm{tr}(\pi(g), V)$$

2nd argument refers to the vector space G acts on

linear transformation

[matrix w.r.t which basis doesn't matter since
$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A)$]

$\chi_\pi(g)$ depends only on the conjugacy class of g, $\mathcal{C}_g$

If $g' \in \mathcal{C}_g$, then

$$\begin{aligned}\chi_\pi(g') &= \chi_\pi(\pi(hgh^{-1})) \text{ for some } h \in G\\ &= \chi_\pi(\pi(h)\pi(g)\pi(h)^{-1})\\ &= \mathrm{tr}(\pi(h)\pi(g)\pi(h)^{-1})\\ &= \mathrm{tr}(\pi(g))\\ &= \chi_\pi(g)\end{aligned}$$

EG Character table of S_3

$$\chi_\pi(g)$$

irreducible rep

$\mathcal{C}_g$

	$\mathcal{C}_1$	$\mathcal{C}_2$	----
π_1			
π_2			
$\vdots$			

What are the conjugacy classes in S_3?

$S_3 = \{Id, (12), (23), (13), (132), (123)\}$

- $\mathcal{C}_{Id} = \{Id\}$

- $\mathcal{C}_{(12)} = \{(12), (13), (23)\} = \mathcal{C}_{(23)} = \mathcal{C}_{(13)}$

  $(1)(12)(1) = (12)$

  $(12)(12)(12) = (12)$

  $(13)(12)(13) = (23)$

  $(23)(12)(23) = (13)$

  $\vdots$

- $\mathcal{C}_{(132)} = \{(123), (132)\} = \mathcal{C}_{(123)}$

What are the irreducible representations of S_3?

- $V = \mathbb{C}$

  $\mathbb{1} : g \mapsto +1$

- $V = \mathbb{C}$

  $\varepsilon_3 : g \mapsto +1$ even permutation

  -1 odd permutation

  parity of permutation = parity of inversion

  τ if $x < y$ but position$(x) >$ position(y)

- $V = W_2 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$

  $St_3 : g \mapsto$ permutation

(These are the only irreducible representations.)

Compute the character table

$\mathcal{C}$	Id	(12)	(123)
$\mathbb{1}$	1	1	1
ε_3	1	-1	1
St_3	2	0	-1

$\mathbb{1}$

$[1] \quad tr(g) = 1$

ε_3

even: $[1] \quad tr(g) = 1$

odd: $[-1] \quad tr(g) = -1$

St_3

Pick a basis of W_2, $B = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ with b_1, b_2

Id: $b_1 \mapsto b_1$, $b_2 \mapsto b_2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad tr = 2$

(12): $b_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = b_2 \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad tr = 0$

$b_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = b_1$

(123): $b_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -b_1 + b_2$

$b_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -b_1$

$\begin{bmatrix} -1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 \end{bmatrix} \qquad \begin{bmatrix} i \end{bmatrix}^{-}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad tr = -1$$

Character is useful!

eg.

- Are U, U' isomorphic?
- Is U irreducible?

First Orthogonality relation

Let (U, π) and (U', π') be two irreducible representations of G.

Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_\pi(g) \overline{\chi_{\pi'}(g)} = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \text{otherwise.} \end{cases}$$

Proof

For any representation U of G,

we write U^G for the space of G-invariant vectors

ie. $U^G = \{ u \in U \mid gu = u \}$

$$\boxed{\text{Claim: } \dim(U^G) = |G|^{-1} \sum_{g \in G} tr(g, U)}$$

The projection from U to U^G is

$$P(v) = |G|^{-1} \sum_{g \in G} gv$$

since

- If $v \in U$, then $P(v) \in U^G$
  ie If $h \in G$, then $h\,P(v) = P(v)$

$$\left[\begin{aligned} hP(v) = h\left(|G|^{-1} \sum_{g \in G} gv\right) = |G|^{-1} \sum_{g \in G} h(gv) &= |G|^{-1} \sum_{hg \in G} (hg)v \\ &= |G|^{-1} \sum_{g \in G} gv \\ &= P(v) \end{aligned} \right]$$

- If $w \in U^G$, then $P(w) = w$

$$\left[P(w) = |G|^{-1} \sum_{g \in G} gw \underset{\uparrow}{=} |G|^{-1} \sum_{g \in G} w = w \right]$$

(w is G-invariant ie $gw = w$)

Trace of a projection matrix = dimension of target space

So $\dim(U^G) = tr\left(|G|^{-1} \sum_{g \in G} g\right) = |G|^{-1} \sum_{g \in G} tr(g)$ $\quad (*)$

Recall, $\mathrm{Hom}_{\mathbb{C}}(V, A) \simeq V^* \oplus \cdots \oplus V^*$ and

$$V \otimes_{\mathbb{C}} A \simeq V \oplus \cdots \oplus V$$

Since V and V' are representations of G, then

$$\mathrm{Hom}_{\mathbb{C}}(V', V) \simeq \qquad V'^* \otimes_{\mathbb{C}} V$$

are also representations of G, with

- G-action on $\mathrm{Hom}_{\mathbb{C}}(V', V)$: $(g\phi)(v') = g(\phi(g^{-1}v'))$
- G-action on $V'^* \otimes_{\mathbb{C}} V$: $g(\psi \otimes v) = (g \bullet \psi) \otimes (g \bullet v) = (\psi \circ g^{-1}) \otimes (gv)$

Apply $(*)$ to $\mathrm{Hom}_{\mathbb{C}}(V', V) \simeq V'^* \otimes_{\mathbb{C}} V$

$$\dim\left[\left(\mathrm{Hom}_{\mathbb{C}}(V', V)\right)^G\right] = |G|^{-1} \sum_{g \in G} tr\left(g, \mathrm{Hom}_{\mathbb{C}}(V', V)\right)$$

$$= |G|^{-1} \sum_{g \in G} tr\left(g, V'^* \otimes_{\mathbb{C}} V\right)$$

Note that for any $V \otimes_{\mathbb{C}} W$,

$$tr(g, V \otimes W) = tr(g, V)\, tr(g, W)$$

$$\dim\left[\mathrm{Hom}_{\mathbb{C}}(V', V)^G\right] = |G|^{-1} \sum_{g \in G} tr(g, V'^*)\, tr(g, V)$$

$$tr(g, V'^*) = tr(g^{-1}, V')$$

$$= |G|^{-1} \sum_{g \in G} tr(g^{-1}, V')\, tr(g, V)$$

$tr(g^{-1}, V) = \overline{tr(g, V)}$ for any V

$$= |G|^{-1} \sum_{g \in G} \overline{tr(g, V')}\, tr(g, V)$$

$$= |G|^{-1} \sum_{g \in G} tr(g, V)\, \overline{tr(g, V')}$$

$$= |G|^{-1} \sum_{g \in G} \chi_{\pi}(g)\, \overline{\chi_{\pi'}(g)}$$

To show:

$$|G|^{-1} \sum_{g \in G} \chi_{\pi}(g)\, \overline{\chi_{\pi'}(g)} = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \text{otherwise} \end{cases}$$

STP:

$$\dim\left[\left(\mathrm{Hom}_{\mathbb{C}}(V', V)\right)^G\right] = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \end{cases}$$

Recall Shur's Lemma,

$$\dim\left(\mathrm{Hom}_G(V, V')\right) = \begin{cases} 1 & \text{if } V \simeq V' \\ 0 & \text{otherwise} \end{cases}$$

STP: $$(\mathrm{Hom}_{\mathbb{C}}(V',V))^G = \mathrm{Hom}_G(V',V)$$

To show: $\mathrm{Hom}_{\mathbb{C}}(V',V) \subseteq \mathrm{Hom}_G(V',V)$

Let $\phi \in \mathrm{Hom}_{\mathbb{C}}(V',V)^G$.
So $(g\phi)(v') = \phi(v')$
To show: $g[\phi(v')] = \phi(gv')$

$$g[\phi(v')] = g[(g^{-1}\phi)(v')]$$

G-invariant

$$= g[g^{-1}(\phi(gv'))]$$

definition

$$= (gg^{-1})\phi(gv')$$
$$= \phi(gv')$$

To show: $\mathrm{Hom}_G(V',V) \subseteq \mathrm{Hom}_{\mathbb{C}}(V',V)$

Let $\varphi \in \mathrm{Hom}_G(V',V)$
So $g[\varphi(v')] = \varphi(gv')$
To show: $(g\varphi)(v') = \varphi(v')$

$$(g\varphi)(v') = g(\varphi(g^{-1}v'))$$

definition

$$= g(g^{-1}(\varphi(v')))$$

G-equivariant

$$= (gg^{-1})\varphi(v')$$
$$= \varphi(v')$$

So now we have shown that

First Orthogonality relation

Let (V,π) and (V',π') be two irreducible representations of G.
Then
$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_{\pi'}(g)} = \begin{cases} 1 & \text{if } \pi \simeq \pi' \\ 0 & \text{otherwise.}\end{cases}$$