Oct. 8 Mixed geometry

Recall from Ting's talk

$\mathcal{O}_0$ = principal block of cat $\mathcal{O}$

$$\mathcal{O}_0^{mix} \to \mathcal{O}_0$$

Two facts: • $\mathcal{O}_0^{mix}$ is Koszul
• self-dual

Goal: $\mathcal{O}_0^{mix}$ is Koszul via mixed geometry

Idea: $\mathcal{O}_0 \cong \mathrm{Perv}_N\, G/B \cong \mathrm{Perv}_B\, G/N$

Incarnations of the grading

X_R/R = finite extension of $\mathbb{Z}$

? motivic $\mathbb{Q}$-sheaves on X_R $\longrightarrow$ $MHM^{\mathbb{Q}}(X_{\mathbb{C}})$

$\longrightarrow$ $\mathrm{Perv}^{mix}_{\mathbb{Q}_\ell} X_{\mathbb{F}_q}$

...

$Y \downarrow X = pt$ $\rightsquigarrow$ $H^*(Y_{\mathbb{C}}, \mathbb{Q})$ w/ Hodge structure

$H^*(Y_{\mathbb{F}_q}, \mathbb{Q}_\ell) \circlearrowleft \mathrm{Fr}$

I. Fix $\overline{\mathbb{Q}_\ell} \xrightarrow{\cong} \mathbb{C}$ use the norm of $\mathbb{C}$. $V/\overline{\mathbb{Q}_\ell} \circlearrowleft \mathrm{Fr}_{r} \in \mathrm{Gal}(\bar{k}/k)$ $\deg k/\mathbb{F}_q = N$ $wt = \beta$ if each eigenvalue α has norm $N^{\frac{\beta}{2}}$

$$\begin{tikzcd} & \operatorname{Spec} \bar{k} \arrow[dl] \arrow[dr, "\bar{x}"] & \\ \operatorname{Spec} k \arrow[rr, "x"] & & X \end{tikzcd}$$

$\mathrm{Fr}_x \in \mathrm{Gal}(\bar{k}/k)$

• $\mathcal{F}$ a $\overline{\mathbb{Q}_\ell}$-sheaf on X is pure of $wt = \beta$ if $\mathcal{F}_{\bar{x}} \circlearrowleft \mathrm{Fr}_x$ is of wt β $\forall x \in X$

• $\mathcal{F}$ is mixed if $\exists$ finite increasing filtration $W_\bullet \mathcal{F}$ s.t. $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is pure

Rmk a vector space $/\overline{\mathbb{Q}}$ decomposes according to eigenvalues but as sheaves there could be extensions

$D^b_{mix}(X, \overline{\mathbb{Q}_\ell}) \ni \mathcal{F}^\bullet$ means $\underline{H}^i(\mathcal{F})$ is mixed $\forall i$

$wt_{\leq}(\mathcal{F}^\bullet) := \max_i (wt_{\leq} \underline{H}^i(\mathcal{F}^\bullet) - i)$

$\mathcal{F}^\bullet$ is pure if $wt_{\leq}(\mathcal{F}^\bullet) = -wt_{\leq}(\mathbb{D}\mathcal{F}^\bullet)$ defined to be $wt_{\mathcal{F}}$

$\cup$

$\mathrm{Perv}_{mix}(X, \overline{\mathbb{Q}_\ell})$ perverse truncations preserves mixedness.

Thm. Each $\mathcal{F}$ has a filtration $W_i\mathcal{F}$ s.t. $W_i\mathcal{F}/W_{i-1}\mathcal{F}$ is pure & wt decreasing

• Any map $f: \mathcal{F} \to \mathcal{G}$ respect the filtration

6 functor formalism.

II.

<u>Defn</u> • A pure Hodge structure of wt n is a finite decreasing filtration $F^p H_{\mathbb{C}}$ s.t. $\forall p,q$ w/ $p+q=n+1$ $F^pH \cap \overline{F^qH}=0$ & $F^pH \oplus \overline{F^q}H = H$

• A mixed Hodge structure is a finite decreasing filtration $F^p H_{\mathbb{C}}$ & a finite increasing filtration $W_m H$

s.t. $gr^W_n H$ w/ F is pure of wt $=n$.

here $F^p(gr^W_n H)_{\mathbb{C}} = F^p \cap W_n \mathbb{C} + W_{n-1}\mathbb{C} / W_{n-1}\mathbb{C}$

• A polarized pure Hodge structure of wt n $((H, H^{p,q}), Q)$ — Q = bilinear form on H

s.t. $Q(F^p, F^{n-p+1}) = 0$

$Q(i^{p-q}\varphi, \bar{\varphi}) > 0$ for $\varphi \neq 0$

• A variation of mixed Hodge structure is a locally constant sheaf $\mathcal{H}$ of $\mathbb{Q}$-v.spaces w/ decreasing Hodge-filtration on $\mathcal{H} \otimes_{\mathbb{Q}} \mathcal{O}_X$ & increasing Weight filtration on $\mathcal{H}$

s.t. ① at each pt get a mixed Hodge structure
② globally satisfies Griffiths transversality.

<u>Example</u>

$$\begin{array}{ccc} & & V \times \mathbb{C}^* \\ & & \downarrow \\ \mathfrak{h}=\mathbb{C} & \xrightarrow{\exp(2\pi i \cdot)} & \mathbb{C}^* \end{array}$$

$\mathbb{Z} = \pi_1(\mathbb{C}^*, 1)$

$\nabla = d - \frac{a}{z} dz \quad a \in \mathbb{Q}$

Monodromy $T = \exp(-2\pi i a)$

$N := \log T = -2\pi i a \in \mathrm{End}(V)$.

canonical extension = unique extension w/ residue at 0 has eigenvalues in $[0,1)$.

$N : \mathfrak{h}(1) \to \mathrm{End}(V)$

compatible w/ Hodge structure $\Leftrightarrow$ a strict morphism.

• $V = \mathbb{C}^2 \quad A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \nabla = d - \frac{A}{z} dz$

$N = \begin{pmatrix} 0 & -2\pi i a \\ 0 & 0 \end{pmatrix}$

$V_{\mathbb{Q}} = \mathbb{Q}\{e_1, e_2\}$.

$W_0 = V_{\mathbb{Q}}$
$\cup$
$W_{-2} = \mathbb{Q}\{e\}$
$\cup$
0

In general, Schmid's orbit theorem: weight filtration on nearby cycle = monodromy filtration

A mixed Hodge module has
- underlying perverse sheaf
- underlying filtered $\mathcal{D}$-module
- weight filtration

Example:
- X non-singular, $\dim = n$. on $\omega_X \circlearrowleft \mathcal{D}_X$

$F_{-n-1}\omega_X = 0 \quad F_{-n}\omega_X = \omega_X$

perverse sheaf is $\mathbb{Q}_X[n]$ pure weight $= n$

- $f : X \to Y \qquad f_*\omega_X$

- Every polarizable variation of Hodge structure of wt w-$\dim Z$ on an Zariski open subset of $Z \subseteq X$ extends uniquely to a polarizable Hodge module w/ strict support Z (closed)

i.e. $\mathcal{V}(M) = IC(Z)$

Defn A polarized variation of Hodge structure of wt $= w$ on X is
1. A local system V of f.dim $\mathbb{Q}$-v.spm
2. A holomorphic v.bundle w/ flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1$
3. A Hodge filtration $F^p\mathcal{V}$ by holomorphic subbundles
4. A bilinear form $Q : V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-w)$

At each pt, get a polarized Hodge structure
globally, Griffiths transversality $\nabla(F^p\mathcal{V}) \subseteq \Omega^1_X \otimes F^{p-1}\mathcal{V}$

Basic facts
- If $\tilde{\mathcal{F}}$ is a Hodge sheaf & $\mathcal{F}$ the underlying perverse sheaf Then $H^i(X, \mathcal{F})$ has a Hodge structure
- In general $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2$, $\mathrm{Ext}^j(\mathcal{F}_1, \mathcal{F}_2)$ carry a natural Hodge structure

denote by $\mathrm{Ext}^j(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in \mathcal{H}_F$

Spectral sequence $E_1^{p,q} = H_{\mathcal{H}}^{2p+q}(\mathrm{Ext}^{-p}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)) \Rightarrow \mathrm{Ext}^i_{\mathcal{H}}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$

Absolute Hodge cohomology $M \in \mathcal{H}_F$

$H^i_{\mathcal{H}}(\mathcal{F}) = \mathrm{Ext}^i_{\mathcal{H}_F}(F(0), M)$

Degenerates at E_2 in particular $0 \to H^1_{\mathcal{H}}\,\mathrm{Hom}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \to \mathrm{Ext}^1_{\mathcal{H}}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \to H^0_{\mathcal{H}}\mathrm{Ext}^1 \to 0$

Lemma $\mathrm{Ext}^j(\tilde{\mathcal{I}}_\alpha, \tilde{\mathcal{I}}_\beta) = \begin{cases} 0 & \text{if } j-n_\alpha+n_\beta \text{ odd} \\ \bigoplus F\left(\frac{-j-n_\alpha+n_\beta}{2}\right) & \text{if } j-n_\alpha+n_\beta \text{ even} \end{cases}$

Pf • Formal reason (parity vanishing) $\Rightarrow$ $\mathrm{Ext}^j(\tilde{\mathcal{I}}_\alpha, \tilde{\mathcal{I}}_\beta) \hookrightarrow \mathrm{Hom}(\mathbb{H}^\bullet(X, \tilde{\mathcal{I}}_\alpha), \mathbb{H}^{\bullet+j}(X, \tilde{\mathcal{I}}_\beta))$

• $\mathbb{H}^\bullet(X, \tilde{\mathcal{I}}_\alpha) \hookrightarrow H^\bullet(Y_\alpha, \mathbb{C})$ by decomposition theorem.

Categorical nonsense

Defn ∘ A mixed category is an Artinian cat $\mathcal{M}$ w/ $w_{\mathcal{M}} = w : \mathrm{Irr}\,\mathcal{M} \to \mathbb{Z}$ (a weight function)

s.t. $\mathrm{Ext}^1(M, N) = 0$ if $w(M) \leq w(N)$

An object is pure of weight i iff all irred comp. has weight i

Lemma: Any object $L \in \mathcal{M}$ has a unique finite increasing filtration $W_\bullet = W_\bullet L$

s.t. $gr_i^W L = W_i L / W_{i-1} L$ is pure of weight i

morphisms are compatible w/ $W_\bullet$

∘ Tate twist of degree $d \in \mathbb{Z}$ is an automorphism

$\langle d \rangle : \mathcal{M} \to \mathcal{M}$

$M \mapsto M\langle d \rangle$

s.t. $w(M\langle d \rangle) = w(M) + d$ for all $M \in \mathrm{Irr}\,\mathcal{M}$

Defn $\mathcal{C}$ = Artinian F-category w/ f.dim Hom's.

$\mathcal{M}$ = mixed cat w/ a degree d Tate twist

• A degrading functor is a pair (ν, ε)

where $\nu : \mathcal{M} \to \mathcal{C}$ exact faithful, sending simple obj's to simple obj's

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\langle d \rangle"'] \arrow[dr, "\nu"] & \\ \mathcal{M} \arrow[r, "\nu"'] \arrow[ur, phantom, "\Leftarrow\varepsilon"] & \mathcal{C} \end{tikzcd}$$

consider: a) $\mathrm{Irr}\,\mathcal{M} \xrightarrow[\cong]{\nu, S} \mathrm{Irr}\,\mathcal{C}$

b) $\forall M, N \in \mathcal{M}$, $i \in \mathbb{Z}$ $\nu^i_{M,N} : \bigoplus_n \mathrm{Ext}^i_{\mathcal{M}}(M, N\langle nd \rangle) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{C}}(\nu M, \nu N)$

a)' ν sends irreducible to irreducible & $\mathrm{Irr}\,\mathcal{M}/\langle d\rangle \xrightarrow{1:1} \mathrm{Irr}\,\mathcal{C}$

b)' indecomposable projective —

Lemma

1. a) & b) $\Rightarrow$ a') b')
2. if $\mathcal{C}$ has enough projectives & $\forall L \in \mathrm{Irr}\,\mathcal{C}$, $\mathrm{End}(L) = F$ Then a) & b) $\Leftrightarrow$ a') & b')

Def n A grading on $\mathcal{C}$ is a triple $(\mathcal{M}, \nu, \varepsilon)$ satisfies a) & b)

Pf of Lemma

- $(-)^{0}_{N,M}$ M, N simple
- $(-)^{1}_{P,N}$ P projective
- $(-)^{0}_{P,P}$ P indecomposable

- Only work to do is the lift indecomposable projectives
- $\Leftarrow$ obvious for projective. use resolution for arbitrary thing

Back to koszulity

Recall $A = \bigoplus_{j \geq 0} A_j$ positively graded, A_0 is semi-simple

Koszul means: for any two pure A-modules M & N of wt m & n, $\mathrm{ext}^i_A(M,N) = 0$ unless $i = m-n$

or equivalently $\mathrm{ext}^i_A(A_0, A_0\langle n\rangle) = 0$ unless $i = n$.

$\mathcal{L}_e$ e = closed orbit in $\mathcal{B}$

$\widetilde{P}_e \to \widetilde{\mathcal{L}}_e$ morphism of Hodge sheaves s.t. $P_e \to \mathcal{L}_e$ is a projective cover

$\widetilde{\mathcal{O}}_{\widetilde{P}_e} :=$ the smallest abelian subcat of $\mathcal{O}_{\mathcal{H}}$ containing $\widetilde{P}_e(j)$

i.e. subquotients of direct sums of $\widetilde{P}_e(j)$

Thm $\widetilde{\mathcal{O}}_{\widetilde{P}_e} \to \mathcal{O}$ is a grading on $\mathcal{O}$

Pf: Need enough projectives (injectives would work)

They are of the form $\widetilde{\mathcal{L}}_\alpha \hookrightarrow V^{i}_\alpha(i) \otimes_{\mathbb{C}} \widetilde{P}_e$

Here Soergel $\Rightarrow \mathcal{L}_\alpha \hookrightarrow \underbrace{V^{\bullet}_\alpha \otimes_{\mathbb{C}} P_e}_{\text{injective envelope.}}$

Example sl_2

$$\begin{tikzcd} 0 \arrow[r, bend left, "a"] & 1 \arrow[l, bend left, "b"] \end{tikzcd}$$

$ba = 0$

$P_0 = \mathbb{C}\{e_0, b\}$ $\mathbb{C} \leftarrow \mathbb{C}$ $\quad I_0 = \mathbb{C}\{e_0, a\}$ $\mathbb{C} \to \mathbb{C}$ $= P_1 \otimes_{\mathbb{C}_2} \mathbb{C}$

$\mathbb{C} \rightleftarrows \mathbb{C}^2$ $\quad P_1 = \mathbb{C}\{e_1, a, ab\} \cong I_1 = \mathbb{C}\{e_1, b, ab\}$ $\quad \mathbb{C} \rightleftarrows \mathbb{C}^2$

* Commutative: On Injectives are * of projectives
$\Theta_w P(e) = P(w)$
$\Rightarrow \Theta_w I(e) = I(w)$

COR $\widetilde{\mathcal{O}}_{\widetilde{P}_e}$ is a Koszul category

pf of COR $\mathrm{Irr}\, \widetilde{\mathcal{O}}_{\widetilde{P}_e} = \{ \widetilde{\mathcal{L}}_\alpha(n) \}$ wt $\widetilde{\mathcal{L}}_\alpha(n) = n_\alpha - 2n$

Koszul $\iff$ $\mathrm{Ext}^i_{\widetilde{\mathcal{O}}_{\widetilde{P}_e}}(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b)) = 0$ if $i \neq n_\alpha - n_\beta - 2a + 2b$

$$\begin{tikzcd}
\mathrm{Ext}^i_{\widetilde{\mathcal{O}}_{\widetilde{P}_e}}(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b)) \arrow[d, "\text{defn of grading}"'] \arrow[r] & \mathrm{Ext}^i_{\mathcal{H}}(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b)) \arrow[d] \\
\mathrm{Ext}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) & H^0_{\mathcal{H}}(\mathrm{Ext}^i(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b))) \arrow[l, Rightarrow]
\end{tikzcd}$$

$$H^0_{\mathcal{H}}(\mathrm{Ext}^i(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b))) = 0$$

Key Lemma $\mathrm{End}\, \widetilde{P}_e \cong$ max W-invariant quotient of $\mathrm{Sym}\, \mathfrak{h}_{\mathbb{Q}}(1)$

$\oplus$ Tate $\mathbb{Q}(i)$'s

In particular

$\Rightarrow$ If $\widetilde{\mathcal{F}}$ is a quotient of direct sums of $\widetilde{P}_e$

then $\mathrm{Hom}(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}})$ is a subquotient of sums of $\mathrm{End}(\widetilde{P}_e)(i)$

$\Rightarrow$ $\mathrm{Hom}(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}) = \oplus\, \mathbb{Q}(i)$'s

$\Rightarrow$ $\widetilde{\mathcal{F}} \in \widetilde{\mathcal{O}}_{\widetilde{P}_e} \iff \mathrm{Hom}(\widetilde{\mathcal{F}}, \widetilde{I}_\alpha) = \oplus\, \mathbb{Q}(i)$'s $\forall \alpha$

$\Rightarrow$ $\widetilde{I}_\alpha(i)$ is injective

$\widetilde{\mathcal{F}} \hookrightarrow \widetilde{\mathcal{G}}$ quotient of $\widetilde{P}_e(i)$

subquotient of $\widetilde{P}_e(i)$

$\square$