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Mixed geometry

X space, scheme $\xrightarrow{IH^*}$ Vector space / abelian gp

mixed Geometry $\rightsquigarrow$ ultimately give a motive

" Basic theory" perverse sheaves, D-modules

Put some extra structure

What we get on a point is " the extra structure".

Q: What is the extra structure?

A: Instead of v. space, we put repr. of a group.

classical point of view, G = Galois group

X/F, F finite field, $G = \hat{\mathbb{Z}} = \mathrm{Gal}(\bar{F}/F)$.

F local field, $\mathbb{Q}_p$ $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ ← structure of this group.

Work over $\mathbb{C}$:

The complex version of mixed Hodge structures.

V space V. two filtrations $W_\cdot$ $F^\cdot$ $\bar{F}^\cdot$

increasing decreasing

Induced filtrations $F^\cdot$ & $\bar{F}^\cdot$ are complimentary on $Gr_n^W V$

ie: $\underbrace{F^p \oplus \bar{F}^{n-p}}_{\text{induced filtrations}} = gr_n^W V$

A vector space

$$V = \bigoplus V^{p,q}, \quad N : V^{p,q} \to \bigoplus_{\substack{a<p \\ b<q}} V^{a,b} \quad \text{linear map.}$$

To go back:

$$W_n = \bigoplus_{p+q\leq n} V^{p,q}$$

$$F^p = \exp(N)\Big(\bigoplus_{a\geq p} V^{a,b}\Big), \quad \bar{F}^q = \exp(-N)\Big(\bigoplus_{b\geq q} V^{a,b}\Big).$$

Second description:

$$MHS \iff \mathrm{Rep}\ V \rtimes \mathbb{G}_m \times \mathbb{G}_m$$

Solvable

$$\mathrm{Lie}(V) = \{ v_{a,b} \mid a, b \in \mathbb{Z} \}$$

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is solvable

Note: Cohom dim of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is 2.
Cohom. dim of MHS is 1. } analogy

$\mathrm{Ind}_{\mathbb{G}_m \times \mathbb{G}_m}^{G}(\chi)$ is indecomposible pro-projective.

Claim:? Any subbundle of a p.proj is proj ?

$$H^i_{\mathcal{H}}(M) \overset{\text{def}}{=} \mathrm{Ext}^i_{MHS}(\mathbb{C}(0), M)$$

M: MHS

only non-zero in degrees 0 & 1.

$X \xrightarrow{f} pt$ smooth (proper)

$$R\mathrm{Hom}_{MHM_X}(f^*\mathbb{C}(0), \mathcal{M}) = R\mathrm{Hom}_{MHM}(\mathbb{C}(0), Rf_*\mathcal{M})$$

$\uparrow$

f smooth $\Rightarrow$ $f^*\mathbb{C}(0)$ is of weight 0.

$\mathcal{H}^d(f^*\mathbb{C}(0)) = 0$, unless $d = \dim(X)$.

$\mathcal{O}_X \rightsquigarrow$ weight $d = \dim X$ $[-d]$.

Rmk: DR: $MHM_X \longrightarrow P(X)$

$\mathcal{M} \longmapsto \Omega_X^{\cdot} \otimes \Omega[\dim X]$

Comment.

Gufang explained: how to get a graded enhancement of $\mathcal{O}_0$ from the mixed story.

It's NOT straightforward!

Jantzen Conjecture

We consider category $\mathcal{O}$ or $\mathcal{O}'$.

We write τ for the - of the Chevally involution.

For an sl_2-triple. $\tau(x_\alpha) = y_\alpha$

$\tau(y_\alpha) = x_\alpha$

$\tau(h_\alpha) = h_\alpha$

$M = M_\lambda$ Verma module of h.w λ.

To construct $M^\vee$, we dualize the weight spaces of M & let $\mathfrak{g}$ act on $M^\vee$:

by $(uf)(v) = f(\tau(u)v)$.

Let v_+ be the h.w vector for M,

$v_+^\vee$ h.w vector for $M^\vee$, $(v_+ \, v_+^\vee) = 1$.

We get a natural map $I: M \longrightarrow M^\vee$

$v_+ \longmapsto v_+^\vee$.

If we go to geometry. $j_! \xrightarrow{I} j_*$

The I gives us a contragredient pairing.

$$C: M \otimes M \longrightarrow \mathbb{C}$$

s.t. $\quad C(uv, v') = C(v, \tau(u) v')$ by

$$C(v, v') = I(v)(v').$$

<u>Note that:</u>

Call them M, $M^\vee$.

the underlying vector spaces of M_λ & $M_\lambda^\vee$ can be identified for various λ. as these modules are just abstractly $\cong U(n_-) v_f$

$$I_\lambda : M \longrightarrow M^\vee. \quad \lambda \in \mathfrak{h}^*.$$

This map can have zeros.

Intertwining Operators

<u>Rmk:</u> Some people like it to have poles.

KL: they fix λ. want to understand how the standard modules & irreducibles are related. at λ.

Jantzen: Look at the neighborhood at λ !!!

Similar to looking at $\mathbb{F}_p \leftrightarrow \mathbb{Q}_p \supseteq \mathbb{Z}_p$. $\odot$.

Let's consider two f.d vector spaces E & F.

$A = \mathbb{C}[[t]]$

$K = \mathbb{C}((t))$

$$\begin{array}{ccc} I_t: A \otimes_{\mathbb{C}} E & \hookrightarrow & A \otimes_{\mathbb{C}} F \\ \big\downarrow \otimes_A K & & \end{array}$$

Free A-mod of rank n.

s.t.

get an isom.

We have a basis of $A \otimes_{\mathbb{C}} F$ $[m_1, \dots, m_n]$

s.t. $\{\underset{= m_1'}{t^{d_1} m_1}, \dots, \underset{= m_n'}{t^{d_n} m_n}\}$ is a basis of $A \otimes_{\mathbb{C}} E$. $d_1 | d_2 | \cdots | d_n$.

Can define descending filtrations F^v, E^v, s.t.:

$$\left.\begin{aligned} A \underset{\mathbb{C}}{\otimes} E^v &= \langle m_i^v \mid d_i \geq v \} \\ A \underset{\mathbb{C}}{\otimes} F^v &= \langle m_i \mid d_i \leq -v \} \end{aligned}\right\} \text{ filtrations on } F \,\&\, E$$

$$E^v / E^{v+1} \cong F^{-v} / F^{-v+1} \quad \text{at } 0.$$

$$m \longmapsto t^{-m} I(m)$$