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From Ting's talk

$\mathcal{O}_0$ = principal block

$$\begin{array}{c} \mathcal{O}_0^{mix} \\ \downarrow \\ \mathcal{O}_0 \end{array}$$

Fact: ① $\mathcal{O}_0^{mix}$ is koszul

② Self dual.

Goal: prove ① via mixed geometry

Idea:

$$\mathcal{O}_0 \cong \mathrm{Per}_N(G/B) \simeq \mathrm{Per}_B(G/N)$$

X_R R = finite extension of $\mathbb{Z}$.

(?) motivic $\mathbb{Q}$-sheaves on X_R $\rightsquigarrow$ $MHM^{\mathbb{Q}}(X_{\mathbb{C}})$

$\rightsquigarrow$ $\mathrm{Per}^{mix}_{\mathbb{Q}_\ell}(X_{\mathbb{F}_p})$.

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \rightsquigarrow H^*(Y_{\mathbb{C}}, \mathbb{Q}) \text{ with Hodge structure.}$$

$$H^*(Y_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \circlearrowleft Fr.$$

Review of MHM:

References
- Schell notes 2014
- Peters - Steenbrink 2008

Take $X = pt$

Def: $H/\mathbb{Q}$ vector space

Pure Hodge structure of wt n: $F^pH_{\mathbb{C}}$ finite decreasing filtration.

s.t: $\forall p, q$ $\quad p+q = n+1$ $\quad F^pH \cap \overline{F^qH} = 0$

$$F^pH \oplus \overline{F^qH} = H.$$

A mixed Hodge str:

- $W_\bullet H$ finite increasing defined $/\mathbb{Q}$.
- $F^pH_{\mathbb{C}}$ $\quad$ s.t: $gr^W_n H$ with $F^\bullet$ pure of wt n.

Rmk: Don't need the $\mathbb{Q}$-structure in reps theory

A variation of ~~mixed~~ pure Hodge module.

is a locally constant sheaf $\mathcal{H}$ of $\mathbb{Q}$-vector spaces

- wt filtration (increasing)
- Hodge filtration $\quad \mathcal{H} \otimes_{\mathbb{Q}} \mathcal{O}_X = \mathcal{V}$

s.t: ① each pt get a pure structure.

② globally $\quad \nabla(F^p \mathcal{V}) \to F^{p-1} \mathcal{V} \otimes \Omega_X^1$.

A mixed Hodge module.

$\mathcal{M} = (M, W, K_{\mathbb{Q}})$

- Underlying perverse sheaf $K_{\mathbb{Q}} = \mathcal{V}(\mathcal{M})$
- M underlying $\mathcal{D}$-mod. filtered.
- W. weight filtration.

s.t: • each strata get VMH

• admissibility.

<u>Example</u>: • X non-singular $\dim X = n$

ω_X pure Hodge module of wt $= n$

$\omega_X \circlearrowleft D_X$ (D-mod str)

$F_{-n} = \omega_X$, $F_{-n-1} = 0$.

Underlying Perverse sheaf $\mathbb{Q}_X[n]$.

• $f: X \to Y$ $\quad f_* \omega_X$ $\quad \alpha(1) =$ polarization.

Notation

$H \leq H_{\mathbb{C}}$.

Twist $H(1) \leq H_{\mathbb{C}}$

$\quad\quad \parallel$

$\quad 2\pi i H$

$F_i H(1) = F_{i-1} H$

<u>Direct image thm</u>:

If X & Y are smooth. f is projective.

<u>Then</u> • $\mathcal{H}^i(f_* \omega_X)$ pure of weight $n+i$.

$\ell: \omega_X \to \omega_X(1)[2]$

Pointwise: c_1 of $\alpha(1)$

$\in H^2(X_y, \mathbb{Z}(1))$.

• $\ell^i: \mathcal{H}^{-i} f_* \omega_X \xrightarrow{\simeq} \mathcal{H}^i f_* \omega_X(i)$.

of Hodge modules.

- If F mixed Hodge mod.

  $H^i(X, v(\mathcal{F}))$ has a Hodge structure.

- In general, $\mathcal{F}_1, \mathcal{F}_2$.

  $\mathrm{Ext}^i_X(v(\mathcal{F}_1), v(\mathcal{F}_2))$ carries a Hodge structure.

$\mathcal{H}_X =$ Cat. of mixed Hodge modules $/X$.

- $\exists$ a spectral sequence: obj in $\mathcal{H}_{\mathbb{Q}} =$ Cat. of mixed structure.

$$E_1^{pq} = H^{2p+q}_{\mathcal{H}}\left(\mathrm{Ext}^{-p}_X(v(F_1), v(F_2))\right) \Rightarrow \mathrm{Ext}^{\cdot}_{\mathcal{H}_X}(F_1, F_2).$$

where: $H^i_{\mathcal{H}}(F) = \mathrm{Ext}^i_{\mathcal{H}_{\mathbb{Q}}}(\mathbb{Q}, F)$

$\uparrow$

$\mathcal{H}_{\mathbb{Q}}.$

degenerate at E_2-page.

In particular,

get a short exact sequence.

$$0 \to H^1_{\mathcal{H}_{\mathbb{Q}}} \mathrm{Hom}_X(\qquad\qquad) \to \mathrm{Ext}^1_{\mathcal{H}_X}(\qquad\qquad\qquad) \to H^0_{\mathcal{H}_{\mathbb{Q}}}(\mathrm{Ext}^1_X(\qquad\qquad)) \to 0.$$

$$\begin{array}{l} \text{open} \\ U \subseteq Z \hookrightarrow X \quad \text{closed} \\ \#\quad\ \ \text{irreducible.} \\ \phi \end{array}$$

Every variation of pure Hodge module on U of wt $= W$ extends uniquely to a pure Hodge of wt W on X with strict support $= Z$

Underly perverse sheaf $= IC$.

<u>Example</u>:

$$\begin{array}{ccc} & & V \times \mathbb{C}^* \\ & & \downarrow \\ \mathfrak{h} & \xrightarrow{\exp(2\pi i\,\cdot)} & \mathbb{C}^* \\ \text{\rotatebox{90}{$=$}} & & \\ \mathbb{C} & & \end{array}$$

$$\nabla = d - \frac{a}{z} dz$$

$T =$ monodromy of $\nabla = \exp(-2\pi i\ a)$

$N = \log T = -2\pi i\, a \in \mathrm{End}(V)(1).$

$$N : \mathfrak{z}(1) \longrightarrow \mathrm{End}(V)$$

① Compatible with the Hodge structure.

② G/N : $N =$ central character of $U(\mathfrak{g})$.

eigenvalues of $N = 0$.

<u>Example:</u>

$$\begin{tikzcd} E \arrow[r] \arrow[d] & 0 \arrow[d, phantom, "\in" description] \\ f : X \arrow[r] & \mathbb{C} \\ X^{\circ} \arrow[r] \arrow[u, hook] & \mathbb{C}^{*} \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

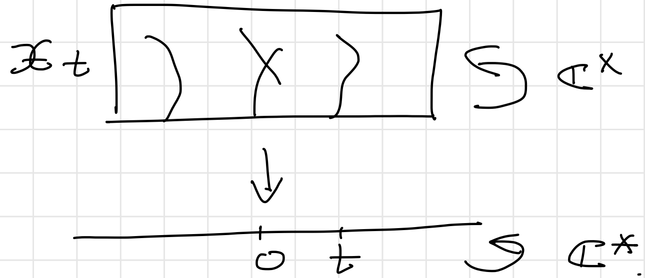
function

E reduced NCD

$$\boxed{\mathcal{H}_f(\mathbb{C}_X) = \Omega^{\bullet}_{X/\mathbb{C}}(\log E).}$$

nearby cycle

topologically.



$\gamma_t : \mathcal{X}_t \longrightarrow \mathcal{X}_0$ some map.

$\gamma_{t*}\mathbb{C}_{\mathcal{X}_t}$

$\circlearrowleft$

T = monodromy.

$\Omega^{\cdot}_{\mathcal{X}/\mathbb{C}}(\log E)$ = relative log deRham Complex. =

order 1 pole along E.

$$= \mathrm{coker}\left(f^*\Omega^1_{\mathbb{C}}(\log 0) \wedge \Omega^{\cdot-1}_{\mathcal{X}}(\log E) \longrightarrow \Omega^{\cdot}_{\mathcal{X}}(\log E) \right)$$

short exact seq:

$$0 \to f^*\Omega^{\bullet}_{\mathbb{C}}(\log 0) \wedge \Omega^{\cdot-1}_{\mathcal{X}/\mathbb{C}}(\log E) \to \Omega^{\cdot}_{\mathcal{X}}(\log E) \to \Omega^{\cdot}_{\mathcal{X}/\mathbb{C}}(\log E) \to 0$$

Thm:

fiber

$$R^q f_* \Omega^{\bullet}_{\mathcal{X}/\mathbb{C}}(\log E)_0 \circlearrowleft \mathrm{Res}_0(\nabla^{GM})$$

GM connection.

$$N = \log(T) = -2\pi i \, \mathrm{Res}_0(\nabla).$$

- Filter by N $\longrightarrow$ weight filtration.

Rmk: In the def. of pure Hodge module of $wt = n$

M is pure of wt n

$\Longleftrightarrow$ $gr^N_\ell(\psi_f M)$ f locally define $wt = n-1+\ell$.

- sl_2-orbit thm: $\exists$ Hodge structure with this weight filtration.

Want to find $\mathcal{O}_0^{mix}$:

$$\downarrow$$

$$\mathcal{O}_0$$

Recall: $A = \bigoplus_{i \geq 0} A_i$ A_0 semi-simple.

Koszul: $\forall$ M, N pure $wt = m, n$ $\left(\begin{array}{l} \text{i.e. } M \text{ is in deg } m. \\ M \circlearrowleft A_0 \end{array} \right)$.
irreducible

$$ext^i_A(M, N) = 0 \quad \text{unless } i = m-n.$$

$X = \mathcal{B} \ni e =$ closed orbit

$\mathcal{L}_e$ irreducible

$\uparrow$ $\quad\quad$ $\in \mathcal{P}er(\mathcal{B})$

$\mathcal{P}_e$ proj. cover.

(graded lifting to the MHM_X)

$\tilde{\mathcal{P}}_e \twoheadrightarrow \tilde{\mathcal{L}}_e$

$\tilde{\mathcal{P}}_e$: (will be proj. in MHM_X).

$\tilde{\mathcal{O}}_{\tilde{\mathcal{P}}_e} =$ the smallest abelian sub of MHM_X containing $\tilde{\mathcal{P}}_e(j)$.

Use: • Each irreducibles. $\to$ Hodge structure.

Thm: $\tilde{\mathcal{O}}_{\tilde{\mathcal{P}}_e} = \mathcal{O}_0^{mix}$ is koszul.

$\mathcal{L}_\alpha$ IC sheaf on a orbit labelled by α.

Ingredients: $\mathrm{Irr}\ \tilde{\mathcal{O}}_{\tilde{\mathcal{P}}_e}$ all of the form $\{\tilde{\mathcal{L}}_\alpha(n)\}$, $wt(\tilde{\mathcal{L}}_\alpha)(n) = n_\alpha - 2n$.

$\alpha, \beta \in W$ $\quad$ $\mathrm{Ext}^i_{\tilde{\mathcal{O}}_{\mathcal{P}e}}(\tilde{\mathcal{L}}_\alpha(a), \tilde{\mathcal{L}}_\beta(b)) = 0$. unless $i = n_\alpha - n_\beta - 2a + 2b$.

Spectral Sequence.

$$\nu(\widetilde{\mathcal{L}}_\alpha) = \mathcal{L}_\alpha$$

$$\begin{tikzcd}
 & \mathrm{Ext}^i_{\widetilde{\mathcal{C}}_{pe}}(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b)) \arrow[dl, "①"'] \arrow[dr] & \\
\mathrm{Ext}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) & & \mathrm{Ext}^i_{MHM_X}(\mathcal{L}_\alpha(a), \mathcal{L}_\beta(b)) \arrow[d] \\
 & & H^0_{\mathcal{H}_{\mathbb{Q}}}(\mathrm{Ext}^i(\mathcal{L}_\alpha\ \mathcal{L}_\beta)) \arrow[ull, hook']
\end{tikzcd}$$

$$\mathrm{Ext}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) \overset{②}{=} 0$$

Suffices to show: ① is inj.

② Vanishing.

Pf of ② Vanishing:

Lemma:
$$\mathrm{Ext}^j(\mathcal{L}_\alpha, \mathcal{L}_\beta) = \begin{cases} 0, & \text{if } j - n_\alpha + n_\beta \text{ odd} \\ \oplus\, \mathbb{Q}\left(\frac{-j - n_\alpha + n_\beta}{2}\right), & \text{if even} \end{cases}$$

Pf:

$$\begin{array}{c} Y_\alpha \\ \downarrow \\ \overline{X}_\alpha \hookrightarrow X \end{array}$$

- $IH^{2i} \hookrightarrow H^{2i}(Y_\alpha, \mathbb{C})$
- vanishes if $i + n_\alpha$ odd.
- $\mathbb{Q}\left(-\frac{i+n_\alpha}{2}\right)$ even.

- $\mathrm{Ext}^j(\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta) \subseteq \mathrm{Hom}\left(IH^{i}(X, \tilde{\mathcal{L}}_\alpha), IH_c^{i+j}(X, \tilde{\mathcal{L}}_\beta)\right)$

$\blacksquare$