Gufang Zhao 8 Oct 2020

From Ting's talk

$\mathcal{O}_0$ = principal block

$$\mathcal{O}_0^{mix} \downarrow \mathcal{O}_0$$

Fact: ① $\mathcal{O}_0^{mix}$ is koszul
② Self dual.

Goal: prove ① via mixed geometry

Idea:

$$\mathcal{O}_0 \cong \mathrm{Per}_N(G/B) \simeq \mathrm{Per}_B(G/N)$$

X_R R = finite extension of $\mathbb{Z}$.

(?) motivic $\mathbb{Q}$-sheaves on X_R $\rightsquigarrow$ $MHM^{\mathbb{Q}}(X_{\mathbb{C}})$

$\rightsquigarrow$ $\mathrm{Perv}^{mix}_{\mathbb{Q}_\ell}(X_{\mathbb{F}_p})$.

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \rightsquigarrow H^*(Y_{\mathbb{C}}, \mathbb{Q}) \text{ with Hodge structure.}$$

$H^*(Y_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \circlearrowleft Fr.$

Review of MHM:

References
- Schell notes 2014
- Peters - Steenbrink 2008

Take $X = pt$

Def: $H/\mathbb{Q}$ vector space

Pure Hodge structure of wt n: $F^p H_{\mathbb{C}}$ finite decreasing filtration.

s.t: $\forall p, q$ $\quad p+q = n+1 \quad F^p H \cap \overline{F^q H} = 0$

$$F^p H \oplus \overline{F^q H} = H.$$

A mixed Hodge str:

- $W_\bullet H$ finite increasing defined $/\mathbb{Q}$.
- $F^p H_{\mathbb{C}}$ s.t: $gr^W_n H$ with $F^\bullet$; pure of wt n.

Rmk: Don't need the $\mathbb{Q}$-structure in reps theory

A variation of ~~mixed~~ pure Hodge module.

is a locally constant sheaf $\mathcal{H}$ of $\mathbb{Q}$-vector spaces

- wt filtration (increasing)
- Hodge filtration $\qquad \mathcal{H} \otimes_{\mathbb{Q}} \mathcal{O}_X = \mathcal{V}$

s.t: ① each pt get a pure structure.

② globally $\qquad \nabla(F^p \mathcal{V}) \to F^{p-1} \mathcal{V} \otimes \Omega_X^1$.

A mixed Hodge module.

$\mathcal{M} = (M, W, K_{\mathbb{Q}})$

- Underlying perverse sheaf $K_{\mathbb{Q}} = \mathcal{V}(\mathcal{M})$
- M underlying $\mathcal{D}$-mod. filtered.
- $W.$ weight filtration.

s.t: • each strata get VMH

• admissibility.

Example: • X non-singular $\dim X = n$

ω_X pure Hodge module of wt $= n$

$\omega_X \circlearrowleft D_X$ (D-mod str)

$F_{-n} = \omega_X$, $F_{-n-1} = 0$.

Underlying perverse sheaf $\mathbb{Q}_X[n]$.

• $f: X \to Y$ $\quad f_* \omega_X$ $\quad \mathcal{O}(1) =$ polarization.

Notation

$H \subseteq H_{\mathbb{C}}$.

Twist $H(1) \subseteq H_{\mathbb{C}}$
$$\begin{array}{c} H(1) \\ \| \\ 2\pi i H \end{array}$$

$F_i H(1) = F_{i-1} H$

Direct image thm:

If X & Y are smooth. f is projective.

Then • $\mathcal{H}^i(f_* \omega_X)$ pure of weight $n+i$.

$\ell: \omega_X \to \omega_X(1)[2]$

Pointwise: c_1 of $\mathcal{O}(1)$
$\in H^2(X_y, \mathbb{Z}(1))$.

• $\ell^i: \mathcal{H}^{-i} f_* \omega_X \xrightarrow{\cong} \mathcal{H}^i f_* \omega_X(i)$.
of Hodge modules.

- If F mixed Hodge mod.

  $H^i(X, \nu(\mathcal{F}))$ has a Hodge structure.

- In general, $\mathcal{F}_1, \mathcal{F}_2$.

  $\mathrm{Ext}^i_X(\nu(\mathcal{F}_1), \nu(\mathcal{F}_2))$ carries a Hodge structure.

$\mathcal{H}_X$ = Cat. of mixed Hodge modules $/X$.

- $\exists$ a spectral sequence: obj in $\mathcal{H}_{\mathbb{Q}}$ = Cat. of mixed structure.

$$E_1^{pq} = H^{2p+q}_{\mathcal{H}}\left(\mathrm{Ext}^{-p}_X(\nu(F_1), \nu(F_2))\right) \Rightarrow \mathrm{Ext}^{\cdot}_{\mathcal{H}_X}(F_1, F_2).$$

where: $H^i_{\mathcal{H}}(F) = \mathrm{Ext}^i_{\mathcal{H}_{\mathbb{Q}}}(\mathbb{Q}, F)$

$\uparrow$

$\mathcal{H}_{\mathbb{Q}}$.

degenerate at E_2-page.

In particular,

get a Short exact sequence.

$$0 \to H^1_{\mathcal{H}_{\mathbb{Q}}} \mathrm{Hom}_X(\qquad) \to \mathrm{Ext}^1_{\mathcal{H}_X}(\qquad) \to H^0_{\mathcal{H}_{\mathbb{Q}}}(\mathrm{Ext}^1_X(\qquad)) \to 0.$$

$$\underset{\substack{\neq\\ \emptyset}}{\overset{\text{open}}{U}} \subseteq Z \hookrightarrow X \quad \text{closed}$$

irreducible.

Every variation of pure Hodge module on U of wt $= W$ extends uniquely to a pure Hodge of wt W on X with strict support $= Z$

Underly perverse sheaf $= IC$.

Example:

$$\begin{array}{ccc} & & V \times \mathbb{C}^* \\ & & \downarrow \\ \mathfrak{h} & \xrightarrow{\exp(2\pi i a)} & \mathbb{C}^* \\ \shortparallel & & \\ \mathbb{C} & & \end{array}$$

$$\nabla = d - \frac{a}{z} dz$$

$T =$ monodromy of $\nabla = \exp(-2\pi i\, a)$

$N = \log T = -2\pi i\, a \in \mathrm{End}(V)(1).$

$N : \mathfrak{z}(1) \longrightarrow \mathrm{End}(V)$

① Compatible with the Hodge structure.

② G/N : $N =$ central character of $U(\mathfrak{g})$.

eigenvalues of $N = 0$.

<u>Example</u>:

$$\begin{tikzcd} E \arrow[r] \arrow[d, hook] & 0 \arrow[d, phantom, "\cap" description] \\ f: \mathfrak{X} \arrow[r] & \mathbb{C} \\ \mathfrak{X}^\circ \arrow[u, hook] \arrow[r] & \mathbb{C}^* \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

function

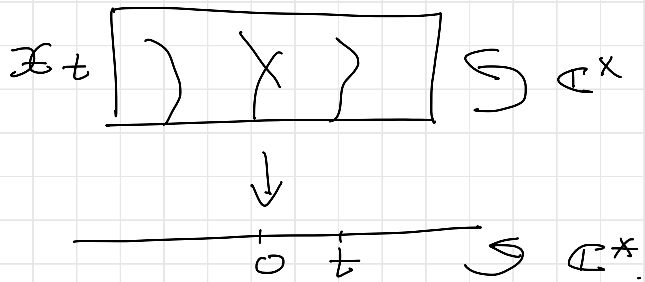
E reduced

NCD

$$\boxed{\mathcal{H}_f(\mathbb{C}_X) = \Omega^\bullet_{\mathfrak{X}/\mathbb{C}}(\log E).}$$

Nearby Cycle

topologically.



$\gamma_t : \mathcal{X}_t \longrightarrow \mathcal{X}_0$ some map.

$$\gamma_{t*}\,\mathbb{C}_{\mathcal{X}_t}$$
$$\circlearrowleft$$
T = monodromy.

$\Omega^\cdot_{\mathcal{X}/\mathbb{C}}(\log E)$ = relative log deRham Complex =

order 1 pole along E.

$$= \operatorname{coker}\left(f^*\Omega^1_{\mathbb{C}}(\log 0) \wedge \Omega^{\cdot-1}_{\mathcal{X}}(\log E) \longrightarrow \Omega^\cdot_{\mathcal{X}}(\log E)\right)$$

short exact seq:

$$0 \to f^*\Omega^\bullet_{\mathbb{C}}(\log 0) \wedge \Omega^{\cdot-1}_{\mathcal{X}/\mathbb{C}}(\log E) \to \Omega^\cdot_{\mathcal{X}}(\log E) \to \Omega^\cdot_{\mathcal{X}/\mathbb{C}}(\log E) \to 0$$

<u>Thm:</u>

$$R^q f_* \Omega^\bullet_{\mathcal{X}/\mathbb{C}}(\log E)_0 \quad \circlearrowleft \quad \operatorname{Res}_0(\nabla^{GM}) \qquad \text{GM connection.}$$

fiber

$$N = \log(T) = -2\pi i \, \mathrm{Res}_0(\nabla).$$

- Filter by $N \longrightarrow$ weight filtration.

Rmk: In the def. of pure Hodge module of wt $= n$

M is pure of wt n

$\Longleftrightarrow$ $\mathrm{gr}^N_\ell(\psi_f M)$ $\quad$ f locally define wt $= n-1+\ell$.

- sl_2-orbit thm: $\exists$ Hodge structure with this weight filtration.

Want to find $\mathcal{O}_0^{mix}$:

$$\begin{array}{c} \mathcal{O}_0^{mix} \\ \downarrow \\ \mathcal{O}_0 \end{array}$$

Recall: $A = \bigoplus_{i \geq 0} A_i$ $\quad$ A_0 semi-simple.

Koszul: $\forall M, N$ pure wt $= m, n$ $\quad$ $\left(\begin{array}{l} \text{ie } M \text{ is in deg } m. \\ M \circlearrowleft A_0 \end{array}\right)$.

irreducible

$$\mathrm{ext}^i_A(M, N) = 0 \quad \text{unless } i = m-n.$$

$X = \mathcal{B} \ni e =$ closed orbit

$\mathcal{L}_e$ irreducible

$\uparrow$

$\mathcal{P}_e$ proj. cover

$\in \mathcal{P}er(\mathcal{B})$.

$\tilde{\mathcal{P}}_e \twoheadrightarrow \tilde{\mathcal{L}}_e$ (graded lifting to the MHM_X)

$\tilde{\mathcal{P}}_e$: (will be proj. in MHM_X).

$\mathcal{O}_{\tilde{\mathcal{P}}_e} =$ the smallest abelian sub of MHM_X containing $\tilde{\mathcal{P}}_e(j)$.

Use: • Each irreducibles $\to$ Hodge structure.

Thm: $\tilde{\mathcal{O}}_{\tilde{\mathcal{P}}_e} = \mathcal{O}_0^{mix}$. is koszul.

$\mathcal{L}_\alpha$ IC sheaf on a orbit labelled by α.

Ingredients: $\mathrm{Irr}\, \tilde{\mathcal{O}}_{\tilde{\mathcal{P}}_e}$ all of the form $\{\tilde{\mathcal{L}}_\alpha(n)\}$, $wt(\tilde{\mathcal{L}}_\alpha)(n) = n_\alpha - 2n$.

$\alpha, \beta \in W$ $\quad \mathrm{Ext}^i_{\tilde{\mathcal{O}}_{\mathcal{P}e}}(\tilde{\mathcal{L}}_\alpha(a), \tilde{\mathcal{L}}_\beta(b)) = 0$. unless $i = n_\alpha - n_\beta - 2a + 2b$.

Spectral Sequence:

$$\begin{tikzcd}
 & \operatorname{Ext}^i_{\widetilde{\mathcal{C}}_{pe}}(\widetilde{\mathcal{L}}_\alpha(a), \widetilde{\mathcal{L}}_\beta(b)) \arrow[dl, "①"'] \arrow[dr] & \\
 & & \operatorname{Ext}^i_{MHM_X}(\mathcal{L}_\alpha(a), \mathcal{L}_\beta(b)) \arrow[d] \\
\operatorname{Ext}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) & & H^0_{\mathcal{H}_{\mathbb{Q}}}(\operatorname{Ext}^i(\mathcal{L}_\alpha\ \mathcal{L}_\beta)) \arrow[ll, hook']
\end{tikzcd}$$

$v(\widetilde{\mathcal{L}}_\alpha) = \mathcal{L}_\alpha$

$\operatorname{Ext}^i(\mathcal{L}_\alpha, \mathcal{L}_\beta) \overset{②}{=} 0$

Suffices to show: ① is inj.

② Vanishing.

Pf of ② Vanishing:

Lemma:
$$\operatorname{Ext}^j(\mathcal{L}_\alpha, \mathcal{L}_\beta) = \begin{cases} 0, & \text{if } j - n_\alpha + n_\beta \text{ odd} \\ \mathbb{Q}\left(\frac{-j - n_\alpha + n_\beta}{2}\right), & \text{if even} \end{cases}$$

Pf :

$$\begin{tikzcd} Y_\alpha \arrow[d] \\ X_\alpha \arrow[r, hook] & X \end{tikzcd}$$

- $\mathbb{H}^{i} \hookrightarrow H^{i}(Y_\alpha, \mathbb{C})$

  $\Downarrow$ vanishes if $i + n_\alpha$ odd.

- $\mathbb{Q}\left(-\frac{i+n_\alpha}{2}\right)$ even.

- $\mathrm{Ext}^{j}(\widetilde{\mathcal{L}_\alpha}, \widetilde{\mathcal{L}_\beta}) \subseteq \mathrm{Hom}(\mathbb{H}^{\cdot}(X, \widetilde{\mathcal{L}_\alpha}), \mathbb{H}_c^{i+j}(X, \widetilde{\mathcal{L}_\beta}))$

$\blacksquare$

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Gufang Zhao.

Def.: • A mixed category is an Artinian $\mathcal{M}$

with:

$$\omega : \mathrm{Irr}\,\mathcal{M} \longrightarrow \mathbb{Z}$$

s.t: $\mathrm{Ext}^1(M, N) = 0$, if $\omega(M) \leq \omega(N)$

• An object is called pure of wt i, if all irr. components have wt $= i$.

Comment: pure objs are all semisimple.

Example: $A = \bigoplus_{i \geq 0} A_i$ f. dim. A_0 is semisimple.

$\mathrm{rad}(A) = A_{>0}$.

$A\text{-}\mathrm{mof}$ = graded f. g modules. ← is an example.

forgetful ↓

$A\text{-}\mathrm{Mof}$ – ungraded f. g modules ← not an example.

A simple mod is a module over A_0 & sitting on a deg ← weight of the module.

Lemma: $L \in M$ any obj.

L has a unique increasing filtration $W_\bullet$

$$gr_i^W(L) \text{ pure of } wt = i.$$

• Tate twist (of deg d).

$\langle d \rangle : \mathcal{M} \longrightarrow \mathcal{M}$ auto-equivalence.

$$M \longmapsto M\langle d \rangle.$$

s.t.: M is irreducible. $w(M\langle d\rangle) = w(M) + d$.

Defn: $\mathcal{C}$ = Artinian. $\mathbb{Q}$-cat with f. dim Hom's.

$\mathcal{M}$ = Mixed cat. with deg d Tate twist.

A degrading functor: (v, ε)

- $v : \mathcal{M} \longrightarrow \mathcal{C}$ exact faithful sending semisimples to semisimples.

- $$\begin{tikzcd} \mathcal{M} \arrow[dr, "v"] \arrow[d, "\langle d\rangle"'] & \\ \mathcal{M} \arrow[r, "v"'] \arrow[ur, phantom, "\Downarrow\varepsilon"] & \mathcal{C} \end{tikzcd}$$

Consider: $v. M \to \mathcal{C}$

a) $\mathrm{Irr}\, M \supseteq S \twoheadrightarrow \mathrm{Irr}\, \mathcal{C}$. (ie any irr. in $\mathcal{C}$ comes from irr (M))

b) $\forall M, N \in \mathcal{M}$.
$i \in \mathbb{Z}$.

$$v^i_{M,N} : \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathcal{M}}(M, N\langle n d\rangle) \xrightarrow{\ \sim\ } \mathrm{Ext}^i_{\mathcal{C}}(vM, vN)$$

isom.

a': ϑ sends irreducibles to irreducibles

& $\mathrm{Irr}(\mathcal{M})/_{\langle d \rangle} \longrightarrow \mathrm{Irr}(\mathcal{C})$.

b') ϑ sends $\begin{pmatrix} \text{indecomp} \\ \& \\ \text{proj.} \end{pmatrix}$ to $\begin{pmatrix} \text{indecomp} \\ \& \\ \text{proj.} \end{pmatrix}$, & $\mathrm{IndProj}(\mathcal{M})/_{\langle d \rangle} \xrightarrow{\cong} \mathrm{IndProj}(\mathcal{C})$

Lemma: ① a) & b) $\Rightarrow$ a') & b').

② If $\mathcal{C}$ has enough proj. & $\forall L \in \mathrm{Irr}(\mathcal{C})$, $\mathrm{End}(L) = \mathbb{Q}$.

Then: a) & b) $\Leftrightarrow$ a') & b').

Defn: A grading on $\mathcal{C}$ is a triple $(\mathcal{M}, \nu, \varepsilon)$ satisfies a) & b).

sketch of the pf of the Lemma.

- For ①: Choose M to be projs ...
- For ②: Using proj resolution

Example:

$\mathcal{H}_{\mathbb{Q}}$ = the cat. of mixed Hodge structures

Obj: Vector spaces $H_{\mathbb{C}}$ +

- $H_{\mathbb{Q}} \subseteq H_{\mathbb{C}}$
- weight filtration $W_\cdot$ (increasing)
- $F^\cdot$: decreasing.
- Polarization: bilinear form.

There will be interesting:

$$\mathrm{Ext}^i_{\mathcal{H}_{\mathbb{Q}}}(\mathbb{Q}(0), V)$$

$\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$ mixed Hodge mods on X.

$$\boxed{\mathrm{Ext}^p_{\mathrm{Perv}_X}(\nu(\tilde{\mathcal{F}}), \nu(\tilde{\mathcal{G}}))}$$ will be V

degrading functor.

$$\nu : \mathrm{MHM}_X \longrightarrow \mathrm{Perv}_X .$$

$X = \mathcal{B}$

$\cup\!\!\!\downarrow$

e closed orbit

$\mathcal{P}_e$ proj. cover.

$\downarrow$

$\mathcal{L}_e$ simple perverse

$\rightsquigarrow$

lift to MHM.

$\tilde{\mathcal{P}}_e$

$\downarrow$

$\tilde{\mathcal{L}}_e$

$\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{P}}_e}$:= the smallest abelian cat contains $\widetilde{\mathfrak{P}}_e(j)$.

$\cap |$

MHM_X.

Thm $\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{P}}_e}$ is Koszul.

Example: $\mathrm{Ext}^1_{H\mathbb{Q}}(\mathbb{Q}(a), \mathbb{Q}(0))$

$$0 \to \mathbb{Q}(0) \to L \xrightarrow{\pi} \mathbb{Q}(a) \to 0.$$ Compatible with all filtrations.

Let $a>0$.

$$L \cong \overset{A}{\overset{\|}{\mathbb{Q}(a)}} \oplus \overset{B}{\overset{\|}{\mathbb{Q}(0)}}$$ underlying vector space.

Wt. filtration: $W_0 L_{\mathbb{Q}} = \mathbb{Q}(0)$

$W_{2a} L_{\mathbb{Q}} = L$.

Hodge filtration. $\left\{F^\bullet L_{\mathbb{C}} \;\middle|\; \begin{array}{l} F^p L \cap A = F^p A \\ \pi F^p L = F^p B \end{array}\right\} / \sim$

$$\left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\} \curvearrowright \{F^\bullet L\}$$ only changes $F^\bullet$.

$* \in \mathbb{C}$

$\Rightarrow$ $\mathrm{Ext}^1_{\mathcal{H}_{\mathbb{Q}}}(\mathbb{Q}(b),\ \mathbb{Q}(a)) = \mathbb{C}/\mathbb{Q}$. (this $\mathbb{Q}$ will gives "$\sim$").

<u>Thm</u>: $\widetilde{\mathcal{O}}_{\widetilde{\mathcal{P}}_e}$ is koszul wt $\widetilde{\mathcal{L}}_\alpha(n) = n_\alpha - 2n$

Need to show: $\mathrm{Ext}^i_{\widetilde{\mathcal{O}}_{\widetilde{\mathcal{P}}_e}}(\widetilde{\mathcal{L}}_\alpha(a),\ \widetilde{\mathcal{L}}_\beta(b)) = 0$ want to show if $i \neq n_\alpha - n_\beta - 2a + 2b$.

$\downarrow$

$\mathrm{Ext}^i_{\mathrm{MHM}_X}(\widetilde{\mathcal{L}}_\alpha(a),\ \widetilde{\mathcal{L}}_\beta(b))$

$\downarrow$ spectral sequence.

(*) suffices to show inj.

$\mathrm{Ext}^i_{\mathrm{Perv}_X}(\mathcal{L}_\alpha,\ \mathcal{L}_\beta) \longleftrightarrow \underbrace{\mathrm{Hom}_{\mathcal{H}_{\mathbb{Q}}}\left(\mathbb{Q}(0),\ \mathrm{Ext}^i_{\mathrm{Perv}_X}(\mathcal{L}_\alpha(a), \mathcal{L}_\beta(b))\right)}$

$\mathrm{Ext}^i_{\mathrm{Perv}_X}(\mathcal{L}_\alpha,\ \mathcal{L}_\beta) = \mathcal{O}_0$

$\mathrm{Ext}^i_{\mathrm{Perv}_X}(\mathcal{L}_\alpha(a), \mathcal{L}_\beta(b)) = \mathrm{Ext}^{i-2b+2a}_{\mathrm{Perv}_X}(\mathcal{L}_\alpha, \mathcal{L}_\beta)$.

gives the wt zero part of $\mathrm{Ext}^i_{\mathcal{O}_0}(\cdots)$.

$\boxed{(a) = \langle a \rangle [2a]}$.

$\boxed{\underline{\text{Lemma}}}$: this wt zero part is zero.

Thm: $\widetilde{\mathcal{O}}_{\widetilde{P}_e} \xrightarrow{\nu} \mathcal{O}_0$ is a grading.

Cor: $\mathrm{Ext}^{i}_{\widetilde{\mathcal{O}}}(\widetilde{L}_\alpha, \widetilde{L}_\beta\langle n\rangle) \cong \mathrm{Ext}^{i}_{\mathcal{O}_0}(L_\alpha, L_\beta)$. (this is condition 6)

$$\begin{array}{c} \mathrm{Ext}^{i}_{\widetilde{\mathcal{O}}}(\widetilde{L}_\alpha, \widetilde{L}_\beta\langle n\rangle) \\ \uparrow \text{inj} \\ \mathrm{Ext}^{i}_{\widetilde{\mathcal{O}}}(\widetilde{L}_\alpha, \widetilde{L}_\beta) \end{array}$$

this implies $(\ast)$

$\boxed{\text{Lemma}}$: the weight zero part is zero:

$$\mathrm{Ext}^{j}(L_\alpha, L_\beta) = \begin{cases} 0 & \text{if } j - n_\alpha + n_\beta \text{ is odd} \\ \oplus\, \mathbb{Q}\left(\frac{j-n_\alpha+n_\beta}{2}\right) & \text{if } j-n_\alpha+n_\beta \text{ even.} \end{cases}$$

Thm: $\widetilde{\mathcal{O}}_{\widetilde{P}_e} \xrightarrow{v} \mathcal{O}_0$ is a degrading.

<u>Pf</u>:

$\mathrm{End}(P_e) = C$ coinvariant alg. $\quad$ $\mathrm{End}(\widehat{\widetilde{P}}_e) = \widetilde{C}$ (has a Hodge str)

$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$

$\mathrm{Sym}(\mathfrak{h}) \circlearrowleft W \qquad\qquad\qquad \mathrm{Sym}(\mathfrak{h}(1))$

Soergel: $\quad L_\alpha \hookrightarrow V_\alpha^\circ \otimes_C P_e = I_\alpha$

$\qquad\qquad \widehat{\widetilde{L}}_\alpha \hookrightarrow V_\alpha^i(i) \otimes \widehat{\widetilde{P}}_e = \widetilde{I}_\alpha$

<u>Example</u>: $\mathfrak{sl}_2$.

$C = C_2 = \mathbb{C}[\varepsilon]/\varepsilon^2 \curvearrowright \mathbb{C}$, ε acts by zero.

$\widetilde{I}_e = \widehat{\widetilde{P}}_e$, and $\widetilde{I}_1 = \widehat{P}_e \otimes_{\widetilde{C}} \mathbb{C}$