Riemann-Hilbert Correspondence

References:

- Gelfand-Manin: Homological algebras
- Arkhipov : Lecture notes D-modules.

I) Hilbert's 21 problem:

For $\forall \rho : \pi_1(\mathbb{C}\mathbb{P}^1 - \{t_1, \cdots t_N\}, *) \longrightarrow GL(n)$,

Is $\rho =$ monodromy reps of

n×n matrix

$$\frac{d\,z(t)}{dt} = A(t)\, z(t)$$

a system of n 1^{st} order ODEs on $\mathbb{P}^1_t$.

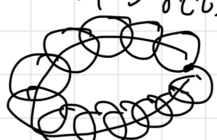
$A(t)\,dt$ has only simple poles contained in $\{t_1, \cdots t_N\}$.

Monodromy:

Thm of ODE: $\mathcal{U}$ simply conn. $\exists$ a unique fundamental solution matrix

$\pi_1 \ni \gamma(t) \quad 0 \le t \le 1$

$$S : \mathcal{U} \longrightarrow \underset{GL_n(\mathbb{C})}{\overset{\subseteq}{M_{n\times n}(\mathbb{C})}}$$



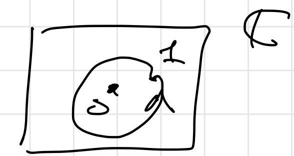
$$\gamma(t) \longmapsto S(\gamma(0))^{-1} \cdot S(\gamma(1))$$

Example: $n=1$

$$\frac{df(z)}{dz} = \frac{a}{z} f(z) \qquad \text{ODE on } \mathbb{P}^1 \setminus \{0, \infty\}$$

(reg. sing at. $0, \infty$)

local solution: $f(z) = c z^a$



If $a \notin \mathbb{Z}$, z^a multivalued function.

Monodromy: $\pi_1(\mathbb{P}^1 \setminus \{0, \infty\}) \longrightarrow GL(1) = \mathbb{C}^*$

$\pi_1(\mathbb{P}^1 \setminus \{0, \infty\}) \cong \mathbb{Z}$

$z = re^{i\theta}$

$1 \longmapsto e^{2\pi i a} =$ difference of r^a & $(r^a e^{i2\pi a})$

Generalization:

- $\mathbb{CP}^1 \rightsquigarrow$ Complex mfd X
- ODE $\rightsquigarrow$
  - × regular singular connection ($\mathcal{O}_X$-coherent D_X-mod) (Deligne 1970)
  - × regular holonomy D-mod. (Kashiwara 1984, Mebkhout 1984)

II) Deligne 1970:

X alg complex mfd.

Def. A connection
- a locally free sheaf of $\mathcal{O}_X$-module $\mathcal{F}$.
- $\mathbb{C}$-linear $\nabla : \mathcal{F} \longrightarrow \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$ s.t

$$\nabla(\varphi f) = d\varphi \otimes f + \varphi \otimes \nabla f \qquad \begin{matrix} \varphi \in \mathcal{O}_X(U) \\ f \in \mathcal{F}(U) \end{matrix}$$

From ∇, one defines:

$$\nabla^{(p)} : \Omega_X^p \underset{\mathcal{O}_X}{\otimes} \mathcal{F} \longrightarrow \Omega_X^{p+1} \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$$

$$\omega \otimes f \longmapsto d\omega \otimes f + (-1)^p \omega \wedge \nabla^{(0)} f$$

∇ is integrable: if $\nabla^{(p+1)} \circ \nabla^{(p)} = 0$ for all p.

Why $\mathcal{D}_X$-mod?

Recall: $\mathcal{D}_X$ is gen. by $\mathcal{O}_X$, & L_ξ, ξ vector field $\in \Omega_X^\vee$.

$[f, g] = 0$, $\quad [L_\xi, f] = \xi(f)$. $\quad [L_\xi, L_\eta] = L_{[\xi, \eta]}$.

Given an $\mathcal{O}_X$-mod $\mathcal{F}$, with $\nabla : \Omega_X^{1\vee} \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$

L_ξ actions via ∇_ξ. $\qquad \xi \longmapsto \nabla_\xi$

Prop: A D_x-mod is $\mathcal{O}_x$-coherent $\Leftrightarrow$ It's locally free of f. fi rank.

X alg complex mfd. $\qquad$ X^{an} analytic mfd.

$p : (X^{an}, \mathcal{O}^{an}) \to (X, \mathcal{O})$ map of ringed spaces

$\mathcal{F} \in \mathrm{QCoh}(X)$. $\qquad$ $\mathcal{F}^{an} := p^* \mathcal{F}$ on X^{an}

Thm A: X^{an} conn. analytic nonsingular mfd. Then, the following cats are equivalent

$(\nabla, \mathcal{F})$ $\qquad$ $\mathrm{Conn}(X^{an}) = \{ \text{Integrable connections on } X^{an} \}$

$\downarrow$ $\qquad$ $\updownarrow$

$\mathcal{E}$ the sheaf of flat sections $\quad$ $LC(X^{an}) = \{ \text{locally constant sheaves of f. dim v. sp.} \}$

$\mathcal{E}(U) = \{ f \in \mathcal{F}(U) \mid \nabla f = 0 \}$ $\qquad$ $\updownarrow$

$\downarrow$ $\qquad$ $\pi_1(X^{an})\text{-mod} = \{ \text{f. dim reps of } X^{an} \}$

Monodromy reps of π_1

$\curvearrowright \mathcal{E}_x, \ x \in X$.

If X is projective (GAGA)

$$(\nabla, \mathcal{F}) \longrightarrow (\nabla^{an}, \mathcal{F}^{an})$$

<u>Thm A'</u>: X proj. alg var.

$$\mathrm{Conn}(X) \xleftrightarrow{\text{Equiv.}} \quad \cdot\ \mathrm{Conn}(X^{an}) \quad \cdot\ LC(X^{an}) \quad \cdot\ \pi_1(X^{an})\text{-mod.}$$

$$\mathrm{Conn}(X) = \{\text{Integrable conn on } X\}$$

<u>Example</u>: For <u>non-cpt</u> mfd. thm A' is wrong.

$X = \mathbb{P}^1 - \{0\}$.

For $\forall$ P polynomial.

$$\begin{array}{c} X \times \mathbb{C} \\ \downarrow \ \uparrow \varphi(z) \\ X \end{array} \quad \nabla_P = d + \frac{P(\frac{1}{z})}{z^2} dz.$$

$$\Longleftrightarrow \frac{d\varphi}{dz} = -\frac{P(\frac{1}{z})}{z^2} \varphi(z).$$

<u>Flat section</u> $\varphi_P(z) = c \exp\left(-\int^z \frac{1}{\xi^2} P(\frac{1}{\xi}) d\xi\right)$

$X^{an} = \mathbb{C} \Rightarrow \pi_1(X^{an}) = \{e\}$

$\longrightarrow$ trivial monodromy.

non-equiv singularities at $z=0$.

Thm B (Deligne 1970)

X smooth conn alg mfd $\qquad$ X^{an} analytic mfd.

Let $\quad \mathrm{Conn}_r(X)$ = the cat. of regular integ. connections on X

$= \{ (\mathcal{F}, \nabla) \}$ conn. with regular sing.

locally free $\mathcal{O}_X$-mod

Then: $\quad \mathrm{Conn}_r(X) \cong$
- $\mathrm{Conn}(X^{an})$
- $LC(X^{an})$
- $\pi_1(X^{an})$-mod.

Def: a) If $X = C$ smooth curve. $\quad C \overset{i}{\hookrightarrow} \bar{C}$ complete.

dense

$(\mathcal{F}, \nabla)$ is regular sing. at C if $\forall p \in \bar{C} \setminus C$. $\quad$ locally $p = \{z=0\}$

$\downarrow$

C

$\nabla = d + A(z)$ order of poles at $p = \{z=0\}$ at most 1.

b) For general X. ∇ is regular sing. if $\forall j : C \hookrightarrow X$. $j^*\nabla$ has regular singular. at C.

III). • Generalization of $LC(X^{an})$.

$Sh(X^{an})$ = the cat. of sheaves of vector spaces on X^{an}

$D^b Sh(X^{an})$ = derived cat of $Sh(X^{an})$

$\cup|$

$D^b_{const}(X^{an}) = \{ \mathcal{F}^\cdot : \quad \to \mathcal{F}^{-1} \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \to \cdots \quad H^i(\mathcal{F}) \text{ is constructible}$

$X^{an} = \bigcup_\alpha S_\alpha \qquad i_\alpha^*(H^i\mathcal{F}) \text{ is locally constant}$

$i_\alpha : S_\alpha \hookrightarrow X^{an} \qquad \text{of f. f. rank} \}$.

• Generalization of $Conn_r(X)$

$D^b(D_X\text{-mod})$

$\cup|$

$D^b_{rh}(D_X\text{-mod})$

regular singularity

M is holonomic if $M = \{0\}$, or $ch(M) \subset T^*X$ is Langrangian.

The de Rham & The solution Functor

$$DR,\ Sol : D^b(D_X\text{-mod}) \longrightarrow D^b(Sh(X^{an}))$$

covariant (DR), contravariant (Sol)

$$DR: M \longmapsto \Omega^{\bullet}_{X^{an}} \otimes_{\mathcal{O}_X^{an}} M^{an}$$

$$= M^{an} \to \Omega^1_X \otimes M^{an} \to \Omega^2_X \otimes M^{an} \to \cdots$$

$$Sol: M \longmapsto R\mathcal{H}om_{D_X^{an}}(M, \mathcal{O}_X^{an}).$$

Why called solution functor?

PDEs $\rightsquigarrow M := D_X \Big/ \sum_{i=1}^{m} D_X P_i$

$P_1, \cdots, P_m$

diff. operators

The space of solutions

$$\mathrm{Hom}(\underset{D_X/I}{\underset{\shortparallel}{M}}, \mathcal{O}_X^{an})$$

$$1 \longmapsto \varphi(1)$$

$$\boxed{P_i(\varphi(1)) = 0.}$$

Prop.:

$$\boxed{Sol(M) = DR(\mathbb{D}(M))[-\dim X].}$$

- $DR(M) = \Omega_X^{\bullet\, an} \otimes_{\mathcal{O}^{an}} M^{an}$

$$= \left(\Omega^{an}_{top} \overset{L}{\otimes}_{\mathcal{D}^{an}} M^{an}\right)[-\dim X] \quad \longleftarrow \quad H^{i}(\Omega_X^{an} \otimes_{\mathcal{O}_X} \mathcal{D}^{an}) = \begin{cases} \Omega^{top} & i = \dim X \\ 0 & \end{cases}$$

- $Sol(M) = \underline{RHom}_{\mathcal{D}_X^{an}}(M, \mathcal{O}_X^{an})$

$$= \mathbb{D}(M) \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{O}_X^{an}.$$

- Exercise: Take $M = \mathcal{D}_X \otimes V$

check the relation

$\boxed{3}$

3 Sep 2020:

Recall

RH for flat connections

Thm B (Deligne)

X smooth conn alg mfd $\qquad X^{an}$ analytic var

Then: $\quad \mathrm{Conn}_r(X) \cong LC(X^{an})$ or $\pi_1(X^{an})$-mod

or $\mathrm{Conn}(X^{an})$

the cat of regular connections on X

(regular sing. at $p \in \overline{X} \setminus X$).

$\overline{X}$ complete.

The deRham & Solution functor.

sheaves of c.v. spaces of X^{an}.

$$DR, Sol : D^b(D_X\text{-mod}) \longrightarrow D^b(Sh(X^{an})\text{-mod})$$

$$DR: \; M \longmapsto \Omega^{\bullet}_{X^{an}} \otimes_{\mathcal{O}_X^{an}} M^{an}$$

$$Sol: \; M \longmapsto R\mathcal{H}om_{D_X^{an}}(M, \mathcal{O}_X^{an}).$$

Prop: $\mathrm{Sol}(M) = \mathrm{DR}(\mathbb{D}(M))[-\dim(X)]$.

Pf: $\mathrm{DR}(M) = \Omega^{\bullet}_{X^{an}} \otimes_{\mathcal{O}_X^{an}} M^{an}$ $\qquad \omega_X^{an} := \Omega^{top}_{X^{an}}$.

$$\overset{*}{=} (\omega_X^{an} \otimes_{\mathcal{D}^{an}} M^{an})$$

Two resolutions of left $\mathcal{D}_X$-mod of $\mathcal{O}_X$

right $\mathcal{D}_X$-mod of ω_X:

$$(*) \quad 0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n \mathrm{Vect}_X \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^1 \mathrm{Vect}_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^0 \mathrm{Vect}_X \to \mathcal{O}_X \to 0$$

$$0 \to \Omega^0_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X \to 0 \quad *$$

gr

$$\begin{array}{c} T^*X \\ \pi \downarrow \uparrow i \\ X \end{array}$$

Koszul complex

$$0 \to \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \Lambda^n \pi^{-1}\mathrm{Vect}_X \to \cdots \to \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \Lambda^0 \pi^{-1}\mathrm{Vect}_X \to i_* \mathcal{O}_X \to 0$$

$$\mathrm{Sol}(M) = R\mathrm{Hom}_{\mathcal{D}_X^{an}}(M, \mathcal{O}_X^{an}) = \mathbb{D}(M) \otimes_{\mathcal{D}_X^{an}} \mathcal{O}_X^{an}.$$

Check the prop for $\mathcal{M} = D_X \otimes V$. $\blacksquare$

Thm C (RH for D-mods).

(a). The de Rham functor

$DR : D^b(D_X\text{-mod}) \longrightarrow D^b(Sh(X^{an}))$ takes $D^b_{hol}(D_X) \to D^b_{const}(X^{an})$.

It induces $D^b_{rh}(D_X\text{-mods}) \xrightarrow{\simeq} D^b_{const}(X^{an})$

constructible complex

(b) On $D^b_{Hol}(D_X)$, DR commutes with $\mathbb{D}$, & $\boxtimes$.

On $D^b_{r.s}(D_X)$, DR commutes with π_*, π^*, $\pi_!$, $\pi^!$.

$\pi : X \to Y$ any map.

(c) DR induces an equivalence

$$RH(D_X) \xrightarrow{\simeq} \mathcal{M}(P_{\frac{1}{2}}, X^{an}, \mathbb{C})$$

(middle perversity?)

regular holo. D-modules

the cat. of perverse sheaves on X^{an}

IV) Example:

$G = SL_2$, $\mathfrak{g} = sl_2$. $\mathbb{P}^1 = G/B$ $= (\mathbb{C}^2 - \{0\})/\mathbb{C}^*$

x, y.

$T_{\mathbb{P}^1} = \mathcal{O}(2)$ tangent sheaf of $\mathbb{P}^1$

Vector fields $\langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}. y\frac{\partial}{\partial y} \rangle$ with equation

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 0$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}(1)^{\oplus 2} \longrightarrow T_{\mathbb{P}^1} \to 0$$

$$\to \{v_1, v_2\} \mapsto v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}$$

$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

For f homog poly of deg k. $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(f) = kf$.

$f(tx) = t^k f(x)$

$$\mathfrak{sl}_2 \longrightarrow \Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}).$$

$$e \longmapsto x\frac{\partial}{\partial y}$$

$$f \longmapsto y\frac{\partial}{\partial x}$$

$$h \longmapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$$[e, f] = h$$

$$\rightsquigarrow \quad U(\mathfrak{sl}_2) \longrightarrow D(\mathbb{P}^1) \quad \Longrightarrow \quad D(\mathbb{P}^1) \cong U(\mathfrak{sl}_2)/\langle C \rangle.$$

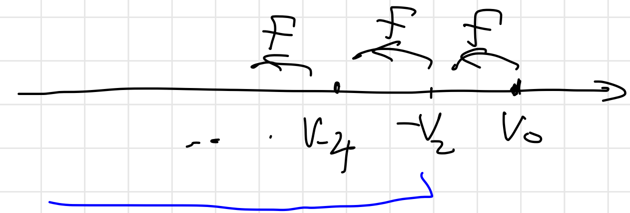
$$C = ef + fe + \frac{h^2}{2} \longmapsto 0.$$

$\mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i}{\hookleftarrow} \infty$

$D_{\mathbb{P}^1}\text{-mod} \longrightarrow \mathrm{Rep}(\mathfrak{sl}_2)$

$\mathcal{O}_{\mathbb{P}^1} \longleftrightarrow \mathbb{C} = L(0)$ trivial.

$j_!(\mathcal{O}_{\mathbb{A}^1}) \longleftrightarrow M_0 = U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_0$



$M_{-2} = U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-2}$

$$\begin{array}{ccccccccc} 0 \to & M_{-2} & \to & M_0 & \to & \mathbb{C} & \to 0 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & U\mathfrak{sl}_2\langle v_{-2}\rangle & & \mathfrak{sl}_2\langle v_0\rangle & & v_0 & \end{array}$$

$\delta_\infty \longleftrightarrow M_{-2}$

$j_*(\mathcal{O}_{\mathbb{A}^1}) \longleftrightarrow M_0^\vee : \quad 0 \to \mathbb{C} \to M_0^\vee \to M(-2) \to$

Perverse sheaves on $\mathbb{P}^1$

$$\mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i}{\hookleftarrow} \{\infty\}$$

- $\mathbb{C}_{\{\infty\}} = \delta_\infty$.
- $IC_{\mathbb{P}^1} = \mathbb{C}_{\mathbb{P}^1}[1]$
- $j_* IC_{\mathbb{A}^1} = j_* \mathbb{C}_{\mathbb{A}^1}[1]$
- $j_! IC_{\mathbb{A}^1} = j_! \mathbb{C}_{\mathbb{A}^1}[1]$

$$i_! i^! \to id \to j_* j^* \xrightarrow{[1]}$$

$$j_! j^! \to id \to i_* i^* \xrightarrow{[1]}$$

$$\Downarrow$$

$$0 \to \mathbb{C} \to M_0^\vee \to M(-2) \to 0$$

$$\boxed{0 \to IC_{\mathbb{P}^1} \to j_* IC_{\mathbb{A}^1} \to \delta_\infty \to 0}$$

$$\boxed{0 \to \delta_\infty \to j_! IC_{\mathbb{A}^1} \to IC_{\mathbb{P}^1} \to 0}$$

$$\downarrow$$

$$0 \to M(-2) \to M(0) \to \mathbb{C} \to$$

(V) The proof of RH (a) by assuming RH (b)

DR commutes with π_*, $\mathbb{D}$, $\boxtimes$, $\pi^!$

<u>Step 1</u>: DR is fully faithful.

<u>Step 2</u>: DR is ess. surjective.

<u>Proof step 1</u>:

$$M, N \in D^b_{rh}(D_X\text{-mod})$$

$$pr : X \to pt$$

$$R\mathrm{Hom}_{D_X}(M,N) \cong pr_* \mathcal{H}om_{D_X}(M,N)$$

$$= pr_* \mathbb{D}(M) \otimes_{\mathcal{O}_X} N$$

$$= pr_* \left(\Delta^! (\mathbb{D}(M) \boxtimes N) \right)$$

$$= DR_{pt}\left(pr_* (\Delta^! (\mathbb{D}(M) \boxtimes N)) \right)$$

$$X \times X \xleftarrow{\Delta} X$$

$$DR_{pt} = id.$$

$$= R pr_* \Delta^! (\mathbb{D}\, DR(M) \boxtimes DR(N))$$

$$= R\Gamma(X^{an}, R\mathcal{H}om_{Sh(X^{an})}(\mathbb{D}DR(M), DR(N)))$$

$$= R\mathrm{Hom}_{Sh(X^{an})}(\mathbb{D}DR(M), DR(N)).$$

<u>Proof of Step 2:</u>

$\forall F \in D^b_{const}(X^{an})$, $\exists M \in D^b_{rh}(\mathcal{D}_X)$ s.t. $DR(M) = F$

Enough to check on generators,

locally closed.

Assume: $F = Ri_* L \in D^b_{const}(X^{an})$, where: $i: Z \hookrightarrow X$

L local system on Z

<u>By Deligne's Thm</u>,

$\exists!$ regular connection $\begin{matrix} N \\ \downarrow \\ Z \end{matrix}$ s.t. $DR_Z(N) = L[\dim Z$

Let $M = i_* N[-\dim Z]$.

Then, $\quad DR(M) = DR(i_* N[-\dim Z])$

$$= Ri_*(DR(N)[-\dim Z])$$

$$= Ri_* \mathcal{L}$$

$$= F.$$