9 Sep 2020

Clarification

(I) algebraic vs analytic D-modules.

Example: $\mathbb{C}\setminus\{0\} \hookrightarrow \mathbb{C}$
$\mathcal{U} = \mathbb{C}\setminus\{0\}$

Two algebraic D-mods on $\mathcal{U}$:

$M = D_{\mathcal{U}}/D_{\mathcal{U}}\partial_z = \mathcal{O}_{\mathcal{U}}$ — integrable connection, $\nabla = d$ (regular sing)

$N = D_{\mathcal{U}}/D_{\mathcal{U}}(z^2\partial_z - 1)$ — $\nabla = d - \frac{1}{z^2}$ (irregular sing)

$M \not\cong N$.

But

$M^{an} \cong N^{an}$

$p \mapsto p\exp(\frac{1}{z})$, where $\exp(\frac{1}{z}) \in \mathcal{O}^{an}$.

$\frac{\partial p}{\partial z} = 0$

$$\frac{\partial(p\exp\frac{1}{z})}{\partial z} = p\cdot(\exp\tfrac{1}{z})\cdot(-1)z^{-2} = -\frac{1}{z^2}p(\exp\tfrac{1}{z}).$$

When defining the dR-functor:

$DR_X : D^b(D_X\text{-mod}) \to D^b(Sh(X^{an}))$

$M \longmapsto \Omega^\bullet_{X^{an}} \otimes_{\mathcal{O}^{an}} M^{an}$

Sol: $M \longmapsto R\mathcal{H}om_{D^{an}}(M, \mathcal{O}_X^{an})$

It's not a good idea to use: $\Omega_X^\bullet \otimes_{\mathcal{O}} M = DR^{alg}$

$R\mathcal{H}om_D(M, \mathcal{O}_X) = Sol^{alg}$

Example:

$X = \mathbb{C}$, $M = D_X/D_X(\frac{d}{dz} - \lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{Z}$.

$\frac{d f(z)}{dz} = \lambda f(z)$ has a solution.

$f(z) = \exp(\lambda z) \in \mathcal{O}_{X^{an}}$ $\rightsquigarrow$ $Sol_X(M) \cong \mathbb{C}_{X^{an}}$, $DR_X(M) = \mathbb{C}_X[1]$

$\exp(\lambda z) \notin \mathcal{O}_{X^{alg}}$ $\rightsquigarrow$ $Sol^{alg}(M) = 0$, $DR_X^{alg}(M) = 0$.

(II) Statement of RH for reg. hol. D-modules (Corrected!!!)

Thm: $DR: D^b(D_X\text{-mod}) \to D^b(Sh(X^{an}))$

(a) It takes $D^b_{hol}(D_X) \to D^b_{const}(X^{an})$ (not fully faithful)

& induces: $D^b_{r.hol}(D_X) \xrightarrow{\simeq} D^b_{const}(X^{an})$

(b) On $D^b_{hol}(D_X)$, DR commutes with $\mathbb{D}$ & $\boxtimes$.

On $D^b_{r.hol}(D_X)$, DR commutes with $\pi_*\ \pi^*\ \pi_!\ \pi^!$

Inverse functor: (Mebkhout)

$D^\infty_{X^{an}}$ = diff. operators of inf. order.

i.e. local coordinate. $x=(x_1,\dots,x_n)$

local section is of the form

$$\sum_{|\alpha|=0}^{\infty} \frac{a_\alpha(x)}{\alpha!} \frac{d^\alpha}{dx^\alpha} \quad \text{with}$$

the condition that:

$$\lim_{|\alpha|\to\infty} |a_\alpha(x)|^{\frac{1}{|\alpha|}} = 0.$$

$= Sol.$

$\mathbb{F}: D_{hol}(D^\infty_X\text{-mod}) \longrightarrow D_{const}(X^{an})$

$M^\infty \longmapsto \mathbb{R}\mathcal{H}om_{D^\infty}(M^\infty, \mathcal{O}_X).$

Inverse:

$\mathbb{G}: D_{const}(X^{an}) \longrightarrow D_{hol}(D^\infty_X\text{-mod})$

$\mathcal{F}^{an} \longmapsto \mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^{an}, \mathcal{O}^{an}_X)$

$\mathbb{F}$ & $\mathbb{G}$ are inverse to each other.

no longer coh. sheaves of $D_{X^{an}}$

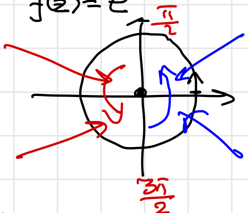
(II)

- For irregular case:

$$\frac{d f(z)}{dz} = \frac{-f(z)}{z^2}$$

Solution: $f(z) = e^{\frac{1}{z}}$ $z = re^{i\theta}$ $\theta = 0 \quad z = r$

$\theta = 2\pi \quad z = r$



Monodromy: is trivial.

But:

$$\lim_{z \to 0} e^{\frac{1}{z}} = \begin{cases} 0, & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \\ \infty, & \frac{3\pi}{2} < \theta < 2\pi \end{cases}$$

$\mathrm{Re}(\frac{1}{z}) < 0$

$|u(z)| = e^{\mathrm{Re}\frac{1}{z}} \to 0$,

$\mathrm{Re}(\frac{1}{z}) > 0$

Two stokes rays: $\frac{\pi}{2}$, $\frac{3\pi}{2}$

Stokes data $\rightsquigarrow$ remembers how the limiting behavior of the solution approaching the singular point.

Have to do with:

$\pi: \mathrm{Bl}_0 X \to X$: local systems on $\mathrm{Bl}_0 X$.

$$\{\text{holonomy } \mathcal{D}\text{-mods}\} \xleftrightarrow{\sim ?} \left\{\begin{array}{l}\text{Perverse sheaves} \\ + \text{ Stokes data}\end{array}\right\}$$

What's Stokes data?

$n \times n$ matrix of formal power series in z

$$\frac{d f(z)}{dz} = \frac{A(z)}{z^{r+1}} f(z)$$

Formal solution:

constant matrix commuting with $Q(z)$

$$H(z) = F(t)\, \underset{\overset{\parallel}{z^{L/r}}}{t^{L}} \exp\left(\underbrace{Q\left(\tfrac{1}{t}\right)}\right), \quad \text{where } t \text{ is a suitable root of } z.$$

$F(z)$ matrix of formal powerseries in z with non-trivial det.

$Q(z)$: diag matrix of polys in z with zero constant term.

Monodromy: $\exp(2\pi i\, L/r) = M$

Stokes rays: $\ell_1 < \ell_2 < \dots \ell_r$ of Q

(on which $e^{Q_i - Q_j}$ has maximal decay)

l_i^+ l_i Stokes ray

l_i^- ~ non-singular ray

H_i^- : fundamental solution along l_i^-

H_i^+ : fundamental solution along l_i^+

<u>Stokes matrix</u> : w.r.t. l_i : $H_i^- = H_i^+ \underline{\underline{S_i}}$

<u>Stokes data</u>:

→ Poincaré rank: r

→ monodromy matrix: M

→ Stokes matrix S_i at l_i