Questions from last time

1) Equivariant D-modules
2) Action of $Z = Z(U(\mathfrak{g}))$. alg grp.

1)// X smooth variety, $X \circlearrowleft H$
$\mu : X \times H \to X$ action map. right action.

$\mathcal{M}$ quasi-coherent sheaf on X.

A left H-action on $\mathcal{M}$ is a map
$\phi : \mu^* \mathcal{M} \longrightarrow pr_X^* \mathcal{M}$
st. identity over $X \times \{1_H\}$
+ cocycle condition.

- $\mathcal{M}$ is a weak (D_X, H)-module if $\mathcal{M}$ is a D_X-module and ϕ is a map of $D_{X\times H/H}$-modules.

(This says action maps $\phi_h : h^* \mathcal{M} \longrightarrow \mathcal{M}$ maps of D_X-modules)

$\mathcal{M}$ is a (D_X, H)-module if ϕ is a map of $D_{X\times H}$-modules.

<u>Claim:</u> ϕ map of $D_{X\times H}$-mods

$$\Updownarrow$$

$\mathfrak{h} = \mathrm{Lie}(H)$-actions on $\mathcal{M}$ given by $i_{\mathfrak{h}}: \mathfrak{h} \to D_X$ and derivative of H-action agree.

<u>Note:</u> $D_{X\times H}$ generated by $D_{X\times H/H}$ and $\mathfrak{h} = \left\{ \begin{array}{c} \text{left invariant} \\ \text{vector fields} \\ \text{on } H \end{array} \right\}$.

<u>$D_{X\times H}$-action on $\mu^*\mathcal{M}$:</u>

$$\mu^*\mathcal{M} = \mu^{-1}\mathcal{M} \otimes_{\mu^{-1}\mathcal{O}_X} \mathcal{O}_{X\times H}$$

Vector fields ∂ on $X\times H$ act by

$$\partial(m\otimes f) = m\otimes \partial f + \mu_*(\partial)(m) f$$

where $\mu_*(\partial) \in \mu^*\mathcal{T}_X = \mu^{-1}\mathcal{T}_X \otimes_{\mu^{-1}\mathcal{O}_X} \mathcal{O}_{X\times H}$ is the pushforward of ∂.

By construction of $i_{\mathfrak{h}}$,

$\mu_*(h) = i_{\mathfrak{h}}(h) \otimes 1$.

So $\mathfrak{h}$-action on $\mu^* M$ is

$$h(m \otimes f) = m \otimes h \cdot f + i_{\mathfrak{h}}(h) m \otimes f.$$

Similarly, $pr_X^* M = pr_X^{-1} M \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times H}$

with $\mathfrak{h}$-action

$$h(m \otimes f) = m \otimes hf.$$

So ϕ map of $D_{X \times H}$-modules iff

$$h \phi(m \otimes 1) = \phi(i_{\mathfrak{h}}(h) m \otimes 1)$$

for $m \in \mu^{-1} M$, $h \in \mathfrak{h}$.

<u>Derivative of H-action on M?</u>

$$\phi(-\otimes 1) \quad \mu^{-1} M \longrightarrow pr_X^{-1} M \otimes \mathcal{O}_H \quad \text{coaction.}$$

$$m \longmapsto (h \longmapsto hm)$$

To get derivative, act by $h \in \mathfrak{h}$ on $\mathcal{O}_H$ then evaluate at $1 \in H$:

$$h \cdot m := (1 \otimes ev_1)\, h \phi(m \otimes 1)$$

So actions agree iff

$$(1 \otimes ev_1)\, h \phi(m \otimes 1) = i_{\mathfrak{h}}(h) m.$$

$\Rightarrow$:

$$(1\otimes ev_1)\, h\, \phi(m\otimes 1) = (1\otimes ev_1)\, \phi(i_{\mathfrak{h}}(h)\, m\otimes 1)$$
$$= i_{\mathfrak{h}}(h)\, m.$$

since coaction

$\Leftarrow$: For this direction, note that coaction

$$\phi : \mu^{-1}\mathfrak{M} \longrightarrow pr_X^{-1}\mathfrak{M}\otimes \mathcal{O}_H$$

is H-equivariant with respect to usual action on $\mathfrak{M}$ on left and right translations on $\mathcal{O}_H$ on right.

$\therefore$ Always have

derivative action

$$h\,\phi(m\otimes 1) = \phi(h\cdot m\otimes 1) \quad \text{for } h\in\mathfrak{h}$$
$$m\in \mu^{-1}\mathfrak{M}.$$

So $h\cdot m = i_{\mathfrak{h}}(h)\, m \Rightarrow$

$$h\phi(m\otimes 1) = \phi(i_{\mathfrak{h}}(h)\, m\otimes 1). \qquad \square$$

2) Action of $Z = Z(U(\mathfrak{g}))$.

We have $\tilde{X} = G \circlearrowright G/N^- \circlearrowleft H$

$\rightsquigarrow U(\mathfrak{g}) \otimes S(\mathfrak{h}) \longrightarrow \Gamma(D_{G/N^-})$

<u>Claim</u>: This factors through

$$U(\mathfrak{g}) \otimes_{Z} S(\mathfrak{h}) \longrightarrow \Gamma(D_{G/N^-})$$

where $Z = Z(U(\mathfrak{g}))$, and

$Z \longrightarrow S(\mathfrak{h})$ is

$$Z \xrightarrow[\text{Harish-Chandra}]{\sim} S(\mathfrak{h})^{W} \subseteq S(\mathfrak{h}) \xrightarrow{-1} S(\mathfrak{h}).$$

Harish-Chandra iso:

Universal Verma module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h}) = M^{uni}$$

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[dr, dashed, "\text{factors through}"'] & U(\mathfrak{g}) \arrow[r] & M^{uni} \\
& & U(\mathfrak{h}) \arrow[u, hook]
\end{tikzcd}$$

singular vectors

Fact: If V is any $U(\mathfrak{g})$-module, then

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "HC"'] & \mathrm{End}(V^{\mathfrak{n}_+}) \\ U(\mathfrak{h}) \arrow[ur] & \end{tikzcd}$$

commutes

Cor: For any $U(\mathfrak{g})$-module V, the diagram

$$\begin{tikzcd} Z \arrow[r, "\iota"] \arrow[d, "HC"'] & Z \arrow[dr] & \\ & & \mathrm{End}(V^{\mathfrak{n}_-}) \\ U(\mathfrak{h}) \arrow[r, "-1"] & U(\mathfrak{h}) \arrow[ur] & \end{tikzcd}$$

commutes.

$\iota : Z \to Z$ induced by $-1 : U(\mathfrak{g})^{op} \to U(\mathfrak{g})$.

Pf: Both maps induced by Chevalley involution on $\mathfrak{g}$.

Pf of claim: For $U \subseteq G/N^-$ open, compare actions on $\mathcal{O}(U) = \mathcal{O}(p^{-1}(U))^{N^-}$, $p : G \to G/N^-$ quotient map.

$Z \subseteq U(\mathfrak{g})$ acts by differentiating the left action on $\mathcal{O}(G)$,

$$(g \cdot f)(x) = f(g^{-1}x), \quad x \in G. \quad \left(\begin{array}{c}\text{right}\\ \text{invariant}\\ \text{vector}\\ \text{fields}\end{array}\right)$$

<u>Fact</u>: This action $U(\mathfrak{g}) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ agrees with

$$U(\mathfrak{g}) \otimes \mathcal{O}(G) \xrightarrow{-1_{\mathfrak{g}}} U(\mathfrak{g}) \otimes \mathcal{O}(G)$$

$$\xrightarrow{Ad^{-1}} U(\mathfrak{g}) \otimes \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$$

derivative of $\left(\begin{array}{c}\text{left}\\ \text{invariant}\\ \text{vector}\\ \text{fields}\end{array}\right)$

$$(g * f)(x) = f(xg).$$

where $Ad^{-1} : U(\mathfrak{g}) \otimes \mathcal{O}(G) \circlearrowleft$ sends

$f : G \to U(\mathfrak{g})$ to the map

$$g \longmapsto Ad_g^{-1} f(g).$$

<u>Pf</u>: Chase vector fields through the diagram

$$\begin{tikzcd}
(g,x) \arrow[r, mapsto] & g^{-1}x \\
G \times G \arrow[r] \arrow[d] & G \arrow[d, equal] \\
G \times G \arrow[r] & G \\
(g,x) \arrow[r, mapsto] & xg
\end{tikzcd}$$

$$(g,x) \longmapsto (x^{-1}g^{-1}x, x)$$

So: Z acts on $\mathcal{O}(p^{-1}(U))$ through

$$Z \xrightarrow{i} Z \subseteq U(\mathfrak{g}) \circlearrowright \mathcal{O}(p^{-1}(U))$$

right translation

So on $\mathcal{O}(U) = \mathcal{O}(p^{-1}(U))^{N^-}$, we have

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "HC"'] \arrow[rr, bend right, "\text{left translations}"'] & Z \arrow[r, "\text{right translations}"] & \Gamma(D_{G/N^-}) \subseteq \mathrm{End}(\mathcal{O}(p^{-1}(U))^{N^-}) \\
S(\mathfrak{h}) \arrow[r, "-1"] & S(\mathfrak{h}) \arrow[ur] &
\end{tikzcd}$$

$\square$