Talk 2 (10/9/20):

Issues from last time:

§ Group actions and D-modules

X = smooth variety / $\mathbb{C}$, D_X = sheaf of diff operators

H = algebraic group $\curvearrowright X$.

Then H acts on D_X in the sense that

- for all $h \in H$, given
$$h : \underbrace{h^* D_X}_{\text{pullback as qcoh sheaf on } X \times X} \longrightarrow D_X$$
algebra homomorphism.
- have
$$i_{\mathfrak{h}} : \mathfrak{h} = \mathrm{Lie}(H) \longrightarrow D_X$$
Lie algebra hom s.t.
$$h \cdot \partial = [i_{\mathfrak{h}}(h), \partial] \text{ for } h \in \mathfrak{h}$$
derivative of H-action

Get $i_{\mathfrak{h}}$ by differentiating left H-action on $\mathcal{O}_X$ given by
$$(h \cdot f)(x) = f(h^{-1}x), \quad h \in H, \ f \in \mathcal{O}_X, \ x \in X.$$

<u>Rmk:</u> $i_{\mathfrak{h}}$ is a quantisation of the moment map $\mu : T^*X \longrightarrow \mathfrak{h}^*$.

<u>Def^n</u>: • A weak (D_X, H)-module is a quasi-coherent left D_X-module $\mathcal{M}$, equipped with an action of H:

$$h : h^*\mathcal{M} \longrightarrow \mathcal{M}, \quad h \in H$$

(map of $\mathcal{O}_X$-modules), s.t.

$$\begin{tikzcd}
h^*(D_X \otimes_{\mathcal{O}_X} \mathcal{M}) \arrow[r] \arrow[d, equal] & h^*\mathcal{M} \arrow[dd, "h"] \\
h^*D_X \otimes_{\mathcal{O}_X} h^*\mathcal{M} \arrow[d, "h \otimes h"'] \arrow[r, phantom, "\circlearrowright"] & \\
D_X \otimes \mathcal{M} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

• A (D_X, H)-module is a weak (D_X, H)-module $\mathcal{M}$ such that

$$i_\eta(h)\, m = h \cdot m \quad \text{for all } h \in \mathfrak{h}$$

$\uparrow$ H-action $\qquad$ $\uparrow$ derivative of H-action.

<u>Monodromic D-modules</u>

Now suppose H is a torus and $\pi : \tilde{X} \longrightarrow X$ is a principal H-bundle.

Then

$$(D_{\tilde{X}}, H)\text{-modules} \xrightarrow{\sim} D_X\text{-modules}$$
$$\mathcal{M} \longmapsto (\pi_* \mathcal{M})^H$$
$$\pi^* \mathcal{N} \longleftarrow\!\shortmid \mathcal{N}$$

push forward as $\mathcal{O}_X$-mods

Weak $(D_{\tilde{X}}, H)$-modules := monodromic D-modules on X

$$\xrightarrow{\sim} \tilde{D}\text{-modules on } X$$
$$\mathcal{M} \longmapsto (\pi_* \mathcal{M})^H$$

where $\tilde{D} = (\pi_* D_{\tilde{X}})^H$

$= $ centraliser of $i_{\mathfrak{h}} : \mathfrak{h} \to \pi_* D_{\tilde{X}}$.

so $\tilde{D}$ has $U(\mathfrak{h}) = S(\mathfrak{h}) \subseteq$ centre

and $\underbrace{D_X = \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}}$. ($\mathfrak{h}$ acts by zero on invariants)

Quantisation of $T^*X = \mu^{-1}(0)/_H$.

<u>Example</u>: $X = pt$, $H = \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$
$\tilde{X} = \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[z, z^{-1}]$

The left action on functions is

$t \cdot z = t^{-1} z$.

$h = t\partial_t$ is sent to $-z\partial_z$ under $i_{\mathfrak{h}}$.

Monodromic D-modules

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Graded $\mathbb{C}[z, z^{-1}]$-modules $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$

s.t. $\deg(z) = -1$

+ $h = -z\partial_z : M \longrightarrow M$ degree 0

s.t. $[h, z] = -1$.

$\xrightarrow{\sim}$ $\tilde{D} = \mathbb{C}[h] = S(\mathfrak{h})$-modules

$M \longmapsto M_0$.

$(D_{\tilde{X}}, H)$-modules

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—— " —— + $h \cdot m = \lambda m$ for $m \in M_\lambda$

$\xrightarrow{\sim}$ $\mathbb{C}[h]$-modules on which $h = 0$

$\|$

Vector spaces = D-mod(pt).

<u>Now to flag varieties again:</u>

G reductive alg grp

$\cup$

B Borel, $H = B/N$.

$\cup$

N unipotent radical

Conventions:

1) [Beilinson-Bernstein "Jantzen conjectures"]

$B = B^-$ (negative roots)

Take $X = G/B^-$, $\tilde{X} = G/N^-$, $h \cdot gN^- = gh^{-1}N^-$

Define $\mathcal{L}_\lambda = G \times^{B^-} \mathbb{C}_\lambda$

$= \pi_*(\mathcal{O}_{\tilde{X}} \otimes \mathbb{C}_\lambda)^H$

Then λ dominant $\Leftrightarrow$ $\mathcal{L}_\lambda$ nef

- $H^0(G/B^-, \mathcal{L}_\lambda) = L(\lambda)$
- $\mathcal{D}^\lambda = \tilde{D}/(i_{\mathfrak{h}}(h) + \lambda(h) \mid h \in \mathfrak{h})$ [Mistake in first talk]

$= \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}_{-\lambda}$ acts on $\mathcal{L}_\lambda$.

2) [E.g. Gaitsgory's notes on category $\mathcal{O}$]

$B = B^+$ (positive roots)

$X = G/B^+$, $\tilde{X} = G/N^+$, $h \cdot gN^+ = gh^{-1}N^+$.

Define $\mathcal{L}_\lambda = \pi_*(\mathcal{O}_{\tilde{X}})_\lambda = \pi_*(\mathcal{O}_{\tilde{X}} \otimes \mathbb{C}_{-\lambda})^H$

$= G \times^{B^+} \mathbb{C}_{-\lambda}$.

Then:

- $H^0(G/B^+, \mathcal{L}_\lambda) = L(-w_0\lambda) = L(\lambda)^*$
- $D^\lambda = \tilde{D}/(i_{\mathfrak{h}}(h) - \lambda(h) \mid h \in \mathfrak{h})$

$= \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda$

acts on $\mathcal{L}_\lambda$.

For this talk, let's follow convention 1.

§ Examples of localisation

Let $G = SL_2$, $B^- = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$, $N^- = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$

$G/N^- \xrightarrow{\sim} \mathbb{C}^2 \setminus \{0\}$, $G/B^- \xrightarrow{\sim} \mathbb{P}^1$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} N^- \longmapsto \begin{pmatrix} b \\ d \end{pmatrix} =: \begin{pmatrix} x \\ y \end{pmatrix}$

H-action: $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} tx \\ ty \end{pmatrix}$

$\rightsquigarrow \tilde{h} := i_{\mathfrak{h}}(\underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{h}}) = \underbrace{-x\partial_x - y\partial_y}$.

Opposite sign from last time.

$e := i_{\mathfrak{g}}(e) = -y\partial_x$

$h := i_{\mathfrak{g}}(h) = y\partial_y - x\partial_x$

$f := i_{\mathfrak{g}}(f) = -x\partial_y$.

<u>Charts:</u> $y \neq 0$, $z = \frac{x}{y}$, $\partial_z = y\partial_x$.

$\tilde{D}|_{y\neq 0} \cong \mathbb{C}[z, \tilde{h}, \partial_z]$

$e = -\partial_z$, $h = -(\tilde{h} + 2z\partial_z)$

$f = \tilde{h}z + z^2\partial_z$

- $x \neq 0$, $w = \frac{y}{x}$, $\partial_w = x\partial_y$.

$\tilde{D}|_{x\neq 0} \cong \mathbb{C}[w, \tilde{h}, \partial_w]$

$e = \tilde{h}w + w^2\partial_w$, $h = \tilde{h} + 2w\partial_w$

$f = -\partial_w$. $z = w^{-1}$

Change of coords: $\partial_z = -\tilde{h}w - w^2\partial_w$

<u>D^λ</u>: Set $\tilde{h} = -\lambda$.

Eg. 1 localise the Verma module $M(\lambda)$:

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

$$\Delta(M(\lambda)) = D^\lambda \otimes_{U(\mathfrak{g})} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

$$= D^\lambda \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

generator

$$= \frac{D^\lambda v}{D^\lambda ev + D^\lambda (h-\lambda)v}$$

<u>On charts</u>: $y \neq 0$

$$\Delta(M(\lambda)) = \frac{D^\lambda v}{\partial_z v = 0 = -2z\partial_z v} \cong \mathcal{O}_{\mathbb{A}^1_z}.$$

$v \longmapsto 1$

$\underline{x \neq 0}$

$$\Delta(M(\lambda)) = \frac{\mathcal{D}^{\lambda}_{v}}{\left(\begin{array}{l} w^2\partial_w v = \lambda w v, \\ 2w\partial_w v = 2\lambda v \end{array}\right)} = \frac{\mathcal{D}^{\lambda}_{v}}{w\partial_w v = \lambda v}$$

Eg. (2) localise the Verma module $M(w_0 \cdot \lambda) = M(-\lambda-2)$.

$$\Delta(M(-\lambda-2)) = \frac{\mathcal{D}^{\lambda}_{v}}{\mathcal{D}^{\lambda}e v + \mathcal{D}^{\lambda}(h+\lambda+2)v}$$

$\underline{\text{On charts:}}$ $y \neq 0$

$$\Delta(M(-\lambda-2)) = \frac{\mathcal{D}^{\lambda}_{v}}{\partial_z v = 0 = [(2\lambda+2) - 2z\partial_z]v}$$

$= 0$ unless $\lambda = -1$

bad hyperplane.
(not dominant)

- $x \neq 0$,

$$\Delta(M(-\lambda-2)) = \frac{\mathcal{D}^{\lambda}_{v}}{\begin{array}{l} w^2\partial_w v = \lambda w v \\ 2w\partial_w v + 2v = 0 \end{array}}$$

$$= \frac{\mathcal{D}^{\lambda}_{v}}{wv = 0} \quad \text{unless } \lambda = -1$$

$= \delta$ module at $w = 0$.

General story:

Set $\mathring{X}_w = N^+ w B^- / B^- \xhookrightarrow{j_w} G/B^-$

$$\Delta(M(w\cdot\lambda)) = j_{w!}(\mathcal{O}_{\mathring{X}_w})$$

since $N^+ w B^- / N^- \longrightarrow \mathring{X}_w$ was an N^+-equivariant section.

$$D^\lambda_{\mathring{X}_w} \cong D_{\mathring{X}_w}.$$

Dual Verma of highest wt λ

$$\Delta(M^\vee(w\cdot\lambda)) = j_{w*}(\mathcal{O}_{\mathring{X}_w}),$$

$\Delta(L(w\cdot\lambda))$ = image of

$$j_{w*}(\mathcal{O}_{\mathring{X}_w}) \longrightarrow j_{w!}(\mathcal{O}_{\mathring{X}_w}).$$

Note: These D^λ-modules are N^+-equivariant. In fact

$$\Gamma : \left\{ \begin{array}{c} N^+\text{-equivariant} \\ D^\lambda\text{-modules} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} N^+\text{-integrable} \\ U^\lambda(\mathfrak{g})\text{-modules} \end{array} \right\}.$$

So structure of N^+-integrable $U^\lambda(\mathfrak{g})$-mods should be closely related to N^+-orbits $\mathring{X}_w$ on G/B^-.