Mixed categories, Ext-duality and Representations
(results and conjectures)

by

Alexander Beilinson and Victor Ginsburg

1. Quadratic algebras. Let k be a field and $A = \oplus_{i\geq 0} A_i$ a graded k-algebra. An algebra A is called "quadratic" if the following holds:

(i) $\dim_k A_i < \infty$ for all $i \geq 0$

(ii) A_o is a semi-simple k-algebra

(iii) A is generated by A_o and A_1

(iv) The ideal of relations (among generators of A) is generated by elements of degree 2.

The structure of an algebra A subject to (i)-(iv) can be spelled out as follows. Let $V := A_1$ be a A_o-bi-module and let $TV = A_o \oplus V \otimes_{A_o} V \oplus V \otimes_{A_o} V \otimes_{A_o} V \oplus \ldots$ be its tensor algebra over A_o. Then, $A \simeq TV/\langle W\rangle$ where $\langle W\rangle$ denotes the two-sided ideal of TV, generated by a A_o-sub-bimodule $W \subset V \otimes_{A_o} V$. Conversely, given A_o, a bi-module V and $W \subset V \otimes_{A_o} V$, one can form a quadratic algebra $TV/\langle W\rangle$.

The dual $A^!$ of a quadratic algebra $A = TV/\langle W\rangle$ is defined as follows. The space $V^* := \mathrm{Hom}_k (V, k)$ has a natural A_o-bimodule structure (with the roles of right- and left-hand actions reversed). Let $W^{\perp} \subset V^* \otimes_{A_o} V^* = (V \otimes_{A_o} V)^*$ be the orthogonal complement to W. Set $A^! = TV^*/\langle W^{\perp}\rangle$. This is also a quadratic algebra.

Given a quadratic algebra A we view A_o as a A-module via the augmentation: $A \longrightarrow A/_{\oplus_{i>0} A_i} \cong A_o$.

The bi-graded space $K(A) = A \otimes_{A_o} (A^!)^*$ has a natural structure of a projective left A-module and of injective right $A^!$-module with an augmentation: $K(A) \longrightarrow A_o \longrightarrow 0$. We endow

$K(A)$ with a differential d, taking $A_i \otimes (A_j^!)^*$ to $A_{i+1} \otimes (A_{j-1}^!)^*$ and commuting with the A-$A^!$-action, as follows. Let $e \in \mathrm{Hom}_{A_o}(V, V) = V^* \otimes_k V$ be the identity element. It can be viewed as an element of $A_1^! \otimes_k A_1 \subset A^! \otimes_k A$

Lemma. $e^2 = 0$ in $A^! \otimes_k A$ (= a tensor product of algebras).

Hence, the multiplication by e on the right gives a differential on $A^! \otimes_{A_o} A$. We let d be the differential on $K(A) = \mathrm{Hom}_{\text{right } A\text{-action}} (A^! \otimes_{A_o} A, A) = A \otimes_{A_o} A^!$, defined by $(df)(x) = f(x\cdot e)$, $x \in A^! \otimes A$. The differential A-$A^!$-module $(K(A), d)$ is called the Koszul complex of A.

Here is an alternative explicit construction of $K(A)$ in terms of V and $W \subset V \otimes_{A_o} V$ (recall that $A = TV/\langle W\rangle$). Let

$W^{i,j} = V^{\otimes i} \otimes W \otimes V^{\otimes j} \subset V^{\otimes(i+j+2)}$. Set:

$$K^{i,j} = \frac{W^{i,j} \cap W^{i+1,j-1} \cap \ldots \cap W^{i+j,0}}{(W^{0,i+j} + W^{1,i+j-1} + \ldots + W^{i-2,j+2}) \cap W^{i,j} \cap \ldots \cap W^{i+j,0}}$$

For each n obvious embeddings give rise to a complex:

$$K^{(n)} : 0 \longrightarrow K^{0,n-2} \longrightarrow K^{1,n-3} \longrightarrow \ldots \longrightarrow K^{n-2,0} \longrightarrow 0$$

If $n = 3$, for instance, then we have:

$$K^{(3)} : 0 \longrightarrow (W \otimes V) \cap (V \otimes W) \longrightarrow V \otimes W \longrightarrow V^{\otimes 3}/W \otimes V \longrightarrow$$
$$\longrightarrow V^{\otimes 3}/(W \otimes V + V \otimes W) \longrightarrow 0$$

We also let, by definition, $K^{(0)} : 0 \longrightarrow A_o \longrightarrow 0$ and $K^{(1)} : 0 \longrightarrow V \xrightarrow{\sim} V \longrightarrow 0$. One can check that the Koszul complex $(K(A), d)$ is isomorphic to the direct sum $\oplus_{n\geq 0} K^{(n)}$ of the complexes $K^{(n)}$.

Let $A_o = \oplus_r A^r$ be the direct sum decomposition of A_o into simple algebras. Each A^r has a unique simple left (resp. right) A^r-module. Simple A_o-bimodules are, hence, labelled by couples (r, s). So let $\{V^{r,s}\}$ be the complete collection of simple A_o-bimodules. Recall, further, that a component A_i, $i = 0, 1, 2, \ldots$ of A is a A_o-bimodule. We define the (matrix valued) Poincaré series of A by

$$P_{r,s}(A, t) = \sum_{i=0}^{\infty} t^i \cdot \dim \mathrm{Hom}_{A_o \otimes A_o}(V^{r,s}, A_i)$$

Theorem 1.1. (cf. LN in Math. N1183). The following conditions on a quadratic algebra A are equivalent

(i) $P(A, t) \cdot P(A^!, -t) = 1$ (matrix identity);

(ii) There is a graded resolution: $0 \leftarrow A_o \leftarrow P_o \leftarrow P_1 \leftarrow \ldots$ where P_i is a projective A-module generated by elements of degree $(-i)$;

(iii) The Koszul complex $K(A)$ is exact , i.e. the complexes $K^{(n)}$ are exact for all $n > 0$;

(iv) For each n the subspaces $W^{i,j} \subset V^{\otimes n}$, $(i + j = n-2)$ generate a distributive lattice, that is, there is a base of $V^{\otimes n}$ such that any $W^{i,j}$ is the span of some base vectors;

(v) The algebra $\mathrm{Ext}^*_A(A_o, A_o)$ is generated by $\mathrm{Ext}^1_A(A_o, A_o)$ over A_o $(=\mathrm{Hom}_A(A_o, A_o))$.

Definition 1.2. A quadratic algebra is called formal if the equivalent conditions (i)-(v) hold.

Proposition 1.3. A quadratic algebra is formal iff there is a graded algebra isomorphism: $A^! \xrightarrow{\sim} \mathrm{Ext}^*_A(A_o, A_o)$.

Corollary 1.4. A is formal iff $A^!$ is formal.

Remark. Formal algebras (in the special case $A_o = k$) were first considered by Priddy (1970), who called them Koszul algebras (cf. LN N°1183). We prefer the name "formal" for the following reason:

Proposition 1.5. If A is formal then the differential graded algebra $R\,\mathrm{Hom}_A(A_o, A_o)$ is quasi-isomorphic to the algebra $\mathrm{Ext}^*_A(A_o, A_o)$ with the trivial differential.

The proof of 1.5 is based on weight considerations, applied to minimal models (see Deligne "Thèorie de Hodge III").

2. Some examples of formal algebras. Throughout this section only formal algebras A with $A_o = k$ are considered. Let V denote a finite-dimensional k-vector space and V^*, the dual space.

2.1. $A = TV$, the tensor algebra. $A^! = k \oplus V^*$ (with trivial multiplication).

2.2. $A = SV$, the symmetric algebra. Then $A^! = \Lambda V^*$, the Grassmann algebra. This example was first considered by J.Bernstein, I.Gel'fand and S.Gel'fand in their work on coherent sheaves on $\mathbb{P}^n$. It was the starting point for our present research.

2.3. (after conversation with V.Drinfeld). Let $R : V \otimes V \longrightarrow V \otimes V$ be a constant unitary R-matrix, i.e. a solution to the Yang-Baxter equation: $R^{12} \cdot R^{23} \cdot R^{12} = R^{23} \cdot R^{12} \cdot R^{23}$ together with the unitarity condition: $R^2 =$ identity. Here R^{ij} is an operator in $V^{\otimes 3}$, acting identically on the factor V not named in the superscript.

Let A be the quotient of TV modulo relations: $x \otimes y - R(x \otimes y) = 0$, $(x, y \in V)$. This is a formal algebra (Zamolodchikov algebra), which is an R-analog of SV. The algebra $A^!$ is equal to the quotient of TV modulo relations $x \otimes y + R(x \otimes y) = 0$.

2.4. Let g be a Lie algebra over k, $U(g)$ the enveloping algebra of g and $k = U_o \subset U_1 \subset \ldots$ the standard filtration on $U(g)$. Let $A = \sum t^i \cdot U_i$ be a subalgebra of $U(g) \otimes k[t]$. Then A is a graded quadratic algebra with generators $\tilde{x} = t \cdot x$, $(x \in g)$ and t, subject to the relations:

$$\tilde{x} \cdot \tilde{y} - \tilde{y} \cdot \tilde{x} = t \cdot \widetilde{[x, y]} \quad \text{and} \quad [\tilde{x}, t] = 0$$

A is formal. The algebra $A^!$ is the cross-product of Λg^* with 2-dimensional algebra $\Lambda k = k \oplus k \cdot e$. The cross-product is determined by the following relations: $e \cdot \omega - (-1)^{\deg \omega} \omega \cdot e = d\omega$, $(\omega \in \Lambda g^*)$, where d denotes the Koszul complex differential in Λg^*.

2.5. (M. Kapranov)* Let $k = \mathbb{C}$ and let $Q_1, \ldots, Q_n$ be quadratic forms on V. Suppose that $X = \{Q_1 = 0\} \cap \ldots \cap \{Q_n = 0\}$ is a complete intersection of quadrics, so that $n \leq \dim V$. Then the coordinate ring $k[X]$ is a formal algebra. The al-

*) to appear in the USSR Izvestia

gebra $A^!$ has generators $x \in V$ and also $h_1, \ldots, h_n$ (deg $h_i = 2$) satisfying the relations:

$$x^2 = Q_1(x)\cdot h_1 + \ldots + Q_n(x)\cdot h_n$$

In fact, one can deduce that $[x, h_i] = 0$. If $n = 1$ then $A^!$ is the Clifford algebra.

More generally, let Y be a projective variety over $\mathbb{C}$, let $i : Y \hookrightarrow \mathbb{P}^N$ be an embedding and $\hat{Y} \subset \mathbb{C}^{N+1}$ the corresponding cone over Y. M.Kapranov has recently shown (to appear) that for a given Y one can always find an embedding i in such a way that the algebra $\mathbb{C}[\hat{Y}]$ is formal.

2.6. Let $\{H_i\}$ be a finite collection of hyperplanes in an affine space $\mathbb{A}^N$ over $\mathbb{C}$. Set $X = \mathbb{A}^N \setminus (\bigcup_i H_i)$. Using the work of Orlik-Solomon one can prove

Proposition 2.6. Suppose that $\pi_i(X)$ are torsion-groups for all $i > 1$. Then the cohomology $H^*(X, \mathbb{Q})$ is a formal algebra.

Example. $\{H_i\}$ is a collection of hyperplanes arising from a root system on $\mathbb{A}^N$. Then $\pi_i(X) = 0$ for $i > 1$ so that 2.6 holds.

Remark. Given any finite collection $\{H_i\}$ one can always add to it a finite number of other hyperplanes $\{H_j'\}$ so that $\pi_n(\mathbb{A}^N \setminus (\bigcup H_i \cup H_j')) = 0$ for $n > 1$. The limit case $X = \mathbb{A}^N \setminus (\varinjlim_{H_i} \bigcup H_i)$ is important in studying motivic cohomology.

If 2.6 holds then the algebra $A = H^*(X)$ has generators x_i, one for each hyperplane H_i. The defining relations among them (besides anti-commutativity) are:

$$x_i\cdot x_j + x_j\cdot x_k + x_k\cdot x_i = 0 \quad \text{for every triple } (H_i, H_j, H_k)$$

such that $\dim (H_i \cap H_j \cap H_k) = \dim \mathbb{A}^N - 2$.

Further, $A^! = U(g)$, where g is a graded nilpotent Lie algebra with generators y_i, one for each hyperplane H_i. The relations among then correspond to pairs (H_i, L), where L is a 1-codimensional subspace of H_i such that $L = H_i \cap H_j$ for some H_j. The relation, attached to a flag (H_i, L) is

$[y_i, \sum y_j] = 0$ (the sum of y_j over all H_j containing L)

The Lie algebra g can be interpreted as the Lie algebra of the Lie group, arising from the nilpotent completion of $\pi_1(X)$.

2.7. The Steenrod algebra is formal [Priddy].

3. Mixed categories. Let k be a field and let C denote an abelian k-category (that is a category having k-vector space structure on Hom's) equipped with an increasing filtration W_i, $(i \in \mathbb{Z})$ on objects of C. The category is called mixed, provided the following holds:

(M1) Any morphism $f : A \to B$ (in the category) is strictly compatible with the filtration, i.e. $f(W_iA) = f(A) \cap W_iB$.

(M2) For any $i \in \mathbb{Z}$ the full subcategory of C formed by objects $Gr_i^W A$, $A \in C$ is a semisimple category with finite-dimensional Hom's.

(M3) For any object A there is $n \gg 0$ such that $W_nA = A$; furthermore $\mathrm{Hom}(A, B) = \varprojlim \mathrm{Hom}(A/W_iA, B/W_iB)$.

We'll assume also an additional axiom

(M4) There is an automorphism $T : C \to C$, having the property: $T(W_iA) = W_{i-1}(TA)$.

The functor T should be thought of as a Tate twist. However, it shifts the weight filtration $W.$ by 1 and not by 2, as usual. We let $T^n(A) = A(n)$.

Remark. The first axiom is equivalent to saying that the functor : $A \mapsto Gr^W A$ is exact.

Examples.

3.1. The category of mixed Hodge structures is a mixed category.

3.2. Let Y be an algebraic variety over a finite field and let $M_{mixed}(Y)$ be the abelian category of mixed perverse $\mathbb{Q}_\ell$-sheaves on Y. This is not a mixed category! However, the strictly full subcategory of $M_{mixed}(Y)$, consisting of perverse sheaves A, such that the Frobenius action on $Gr^W A$ is semisimple, is a mixed category.

3.3. Let $A = \oplus_{i \geq 0} A^i$ be a graded k-algebra such that $\dim_k A^i < \infty$ and A^o is a semisimple algebra. Let Mod A be the category of graded A-modules $M = \oplus_{i \in \mathbb{Z}} M^i$ such that:

(i) $\dim M^i < \infty$, $(i \in \mathbb{Z})$; (ii) $M^i = 0$ for $i << 0$. Then Mod A is a mixed category with weight filtration, defined by $W_n M := \oplus_{i \geq -n} M^i$ and Tate twist being the grading shift.

3.4. Let A be as in 3.3. ~~We~~ Let Proj A be the category of complexes: $\ldots \to P_i \to P_{i+1} \to \ldots$, having the properties: (i) P_i is a graded projective A-module, generated by (a finite number of) elements of degree i; (ii) differentials in complexes are compatible with gradings; (iii) $P_i = 0$ for $i >> 0$. Let $W.$ be the "stupid" filtration on complexes. Then (Proj A, W.) turns out to be a mixed category with Tate twist, given by the translation of a complex.

An object M of a mixed category C is called pure of weight n if $W_n M = M$ and $W_{n-1} M = 0$.

Proposition 3.5. Suppose C is an (artinian) mixed category with a finite number of simple objects of weight 0. ~~If C is complete with respect to W.~~, then C is equivalent to the category Mod A (cf. 3.3) of graded modules over a graded algebra $A_C = \oplus_{i \geq 0} A^i_C$, where A^0_C is a semisimple algebra, $\dim A^i_C < \infty$

Proposition 3.6. Let M and N be pure objects of C of weights m and n. Then:

(i) $\mathrm{Ext}^i_C (M, N) = 0$ for $i > m-n$

(ii) $\mathrm{Ext}^{m-n}(M, N)$ is spanned by products of Ext^1's.

Definition 3.7. A mixed category C is called formal, provided the following holds: for any pure objects M and N of weights m and n one has: $\mathrm{Ext}^i_C (M, N) = 0$ for all $i \neq m-n$.

Theorem 3.8. A mixed category Mod A (cf. 3.3) is formal iff the algebra A is formal in the sense of 1.2.

Corollary 3.9. If Mod A is a formal category, then A is a quadratic algebra.

Proposition 3.10. Let M be an object (of a formal category), having a unique simple quotient. Then $W_i M / W_{i-1} M$ is the maximal semisimple quotient of $W_i M$ (for any $i \in \mathbb{Z}$).

Remarks. (i) The only assumption, actually used in the proof of the proposition is: for pure modules K, L of weights k and l $\mathrm{Ext}^1(K, L) = 0$ unless $k - l = 1$.

(ii) The proposition says that the weight filtration on M coincides with the co-socle filtration on M.

(iii) Suppose that $W_k M = 0$ for some $k \in \mathbb{Z}$ and that M has a unique simple submodule. Then the dual statement holds: the weight filtration on M coincides with the socle filtration.

Main theorem 3.11. Let A be a formal algebra. Then the derived categories D^-Mod A and D^-Mod $A^!$ of bounded from above complexes of graded A- and $A^!$-modules are equivalent.

Remarks.

3.12. The equivalence is defined, roughly speaking, via the assignment: $M \mapsto K(A^!) \otimes_A M$ (recall, that the Koszul complex $K(A^!)$ has a right A-module structure). It takes simple A-modules into projective $A^!$ -modules.

3.13. The heart of D^-Mod A, corresponding to a non-standard t-structure arising from the standard one on D^-Mod $A^!$, is the abelian category Proj A (cf. 3.4).

3.14. The theorem can be applied to all examples, listed in § 2. In the case 2.2 it reduces to a BGG-theorem and in the case 2.5 gives the result of M. Kapranov.

3.15. We can, in fact, prove a generalization of 3.11, valid for triangulated mixed categories with t-structures, that are not arising from abelian categories. Such a generalization is essential for applications to affine Hecke algebras, for instance.

4. Applications to $\underline{g}$-modules.

Let $\underline{g}$ be a complex semisimple Lie algebra with Borel subalgebra $\underline{b}$. Let $Z(\underline{g})$ be the center of $U(\underline{g})$, the enveloping algebra, and Z_+ the augmentation ideal of $Z(\underline{g})$. Let O be the category of finitely-generated $U(\underline{g})$-modules M such that: (i) $Z_+ \cdot M = 0$; (ii) the $U(\underline{b})$-action on M is locally finite.

Let W denote the Weyl group of $\underline{g}$, M_w the Verma module with the highest weight $-w \cdot \rho - \rho$ (ρ = half-sum of positive roots), L_w its simple quotient and P_w the unique indecomposable projective cover of L_w. Set $L = \oplus_{w \in W} L_w$ and P =

$= \oplus_{w \in W} P_w$. The category O is equivalent to the category of finite-type modules over $A_O := \mathrm{Hom}_O(P, P)$, a finite dimensional algebra.

Let X be the Flag manifold associated with $\underline{g}$, let $X = \sqcup X_w$ be the Schubert cell stratification of X and let $T^*_{X_w} X$ denote the conormal bundle to X_w. We recall

<u>Theorem 4.1 (Beilinson-Bernstein, Brylinski-Kashiwara)</u>. The category O is equivalent to the category of regular holonomic D_X-modules (resp. perverse sheaves) whose characteristic varieties are contained in $\sqcup T^*_{X_w} X$.

We define a mixed category O_{mixed} using the recent M. Saito's theory of mixed Hodge modules. Specifically, let O_{mixed} be the category of geometric mixed Hodge modules on X whose characteristic varieties are contained in $\bigsqcup_w T^*_{X_w} X$.[1)]

<u>Remark.</u> There is a possibility of defining O_{mixed} using l-adic mixed sheaves on the Flag variety over a finite field. One would then face troubles connected with semisimplicity of the Frobenius action.

Let $F : O_{mixed} \longrightarrow O$ be an obvious functor, forgetting the mixed structure.

<u>Proposition 4.2</u>. The functor F gives rise to an isomorphism:

$$\mathrm{Ext}^*_{O_{mixed}} (M, \oplus_{i \in \mathbb{Z}} N(i)) \xrightarrow{\sim} \mathrm{Ext}^*_O (FM, FN), \quad M,N \in O_{mixed}$$

The objects L_w and M_w have their mixed counterparts $L^m_w, M^m_w \in O_{mixed}$, normalized in such a way that $Gr^W_O M^m_w = L^m_w$. Clearly, $F(L^m_w) = L_w$, $F(M^m_w) = M_w$.

<u>Lemma 4.3</u>. The category O_{mixed} has enough projectives.

Let P^m_w be the indecomposable projective cover of L^m_w. One can show, that $F(P^m_w) = P_w$. By abusing notations we'll identify L^m_w with L_w, etc. ..., dropping the superscript "m" in future. Recall that $P = \oplus P_w$.

Consider the graded algebra $A_{mixed} = \oplus_{i \in \mathbb{Z}} \mathrm{Hom}_{O_{mixed}} (P, P(i))$. The category O_{mixed} is equivalent to $\mathrm{Mod}\, A_{mixed}$. Furthermore, by 4.2, the algebra A_{mixed} is, in fact, isomorphic to A_O.

1) with $M(\frac{1}{2})$ denoting the square root of the ordinary Tate twist, formally added for convenience.

Hence

<u>Corollary 4.4</u>. There is a graded algebra structure on $A_O = \mathrm{Hom}_O (P, P)$.

The degree O component of A_O can be easily described as a commutative algebra $\oplus_{w \in W} \mathbb{C} \cdot id_{P_w}$.

Let $l(\cdot)$ denote the length function on W and let $P_{y,w}(t)$, $(y, w \in W)$ be the Kazhdan-Lusztig polynomials. Consider a $\mathbb{Z}[t]$-valued matrix $Q(t)$ with entries $Q_{y,w}(t) = t^{l(w)-l(y)} \cdot P_{y,w}(t^{-2})$ $(= \sum t^i \cdot \dim \mathrm{Ext}^i_O (M_y, L_w))$.

<u>Theorem 4.5</u>. The graded algebra A_O (hence, the category O_{mixed}) is formal; its Poincaré (matrix valued) polynomial equals:

$$P\,(A_O;\, t) = Q^T(t) \cdot Q(t)$$

<u>Corollary 4.6</u>. $\mathrm{Hom}_O (P, P)$ is a quadratic algebra with $\mu(y, w)$ generators contained in each $\mathrm{Hom}\,(P_y, P_w)$ and the same number of generators in $\mathrm{Hom}(P_w, P_y)$. (Here $\mu(y, w)$ is the leading coefficient of $P_{y,w}(t)$, that is of the power $t^{1/2(l(w)-l(y)-1)}$.)

<u>Corollary 4.7</u>. The weight filtrations on P_w's and M_w's coincide with co-socle filtrations (see 3.10 ; [Ba]).

<u>Corollary 4.8</u>. The modules M_w and L_w have projective resolutions, such that their i-th term is generated by elements of weight $(-i)$ (cf. 1.1 (ii)).

The proof of theorem 4.5 is based on the following two propositions. Let $DR(M)$ denote the perverse sheaf on X, corresponding to $M \in O_{mixed}$.

<u>Proposition 4.9</u>. For any $M,N \in O_{mixed}$:

$$\mathrm{Ext}^i_{O_{mixed}} (M, N) = \text{weight } 0 \text{ part of } H^i(DR(M^*) \overset{!}{\otimes} DR(N))$$

The proposition says, that Ext's in O_{mixed} coincide with those in the ambient category of all mixed Hodge modules on X. Our proof of 4.8 is based on a reduction technique developed in [Be] (and based on a "maximal extension functor" Ξ). Proposition 4.2 follows from 4.9 and from a "non-mixed" version of 4.9 which was established in [GaJ].

Let $\mathcal{I}_w = DR(L_w)$ be the Intersection cohomology complex on X_w. The filtration of X by the strata X_w of dimension $\leqslant i$ gives rise to a spectral sequence:

$$E_2 = \oplus_{w \in W} (H^*_w \mathcal{L}_x \otimes H^*_w \mathcal{L}_y) \Rightarrow H^*(\mathcal{L}_x \overset{!}{\otimes} \mathcal{L}_y) \qquad (4.9)$$

The existence of nice normal slices to Schubert cells yields the following

Proposition 4.10. The spectral sequence (4.9) collapses; moreover,

$$\mathrm{Ext}^n(\mathcal{L}_x, \mathcal{L}_y) = \oplus_{i+j=n,\ w \in W} H^i_w \mathcal{L}_x \otimes H^j_w \mathcal{L}_y$$

is a space of weight n.

It follows from 4.8-4.10 that the algebra A_{mixed} is formal and that the dual algebra $A^!_{mixed} = \oplus_i \mathrm{Ext}^i(L, L(i))$ has the Poincarè polynomial $P(A^!_{mixed}; t) = Q^T \cdot Q$. To get 4.5 we remark that $P(A_0, t) \overset{(\text{see } 1.1(i))}{=} P(A^!_0, -t)^{-1} = Q(-t)^{-1} \cdot Q^T(-t)^{-1}$. Furthermore, the inversion formula [KL, 3.1] for Kazhdan-Lusztig polynomials:

$$\sum_{x \leq z \leq y} (-1)^{l(z)-l(x)} \cdot P_{x,z} \cdot P_{w_0 y,\, w_0 z} = \delta_{x,y}$$

shows that the matrix $Q(-t)^{-1}$ coincides with $Q^T(t)$ up to the permutation of indexes: $x \longmapsto w_0 \cdot x$, where w_0 is the longest element of W. That gives 4.5 as well as the following surprising

Corollary 4.11. $P(A_0; t) = P(A^!_0; t)$

4.12. Let $\mathcal{L} = \oplus_{w \in W} \mathcal{L}_w$ be the direct sum of the Intersection cohomology complexes on all strata X_w. The algebra $A^!_0 = \mathrm{Ext}^*(\mathcal{L}, \mathcal{L}) = H^*(\mathcal{L} \overset{!}{\otimes} \mathcal{L})$ has the following "micro-local" interpretation. Let Δ denote the diagonal of $X \times X$, $T_\Delta(X \times X)$ the normal bundle of Δ in $X \times X$ and $i : \Delta \hookrightarrow T_\Delta(X \times X)$, the zero-section embedding. Let $Sp_\Delta(\mathcal{L} \boxtimes \mathcal{L})$ be the Verdier specialization of $\mathcal{L} \boxtimes \mathcal{L}$, a perverse sheaf on $X \times X$, to Δ. Thus, $Sp_\Delta(\mathcal{L} \boxtimes \mathcal{L})$ is a perverse sheaf on $T_\Delta(X \times X)$ and it is clear, that: $\mathcal{L} \overset{!}{\otimes} \mathcal{L} = i^! \, Sp_\Delta(\mathcal{L} \boxtimes \mathcal{L})$. Next, identify $T_\Delta(X \times X)$ with TX, the tangent bundle on X. Let F be a

self-dual monodromic perverse sheaf on T^*X defined as the Fourier transform of $Sp_\Delta(\mathcal{L} \boxtimes \mathcal{L})$. The complex F is supported on $\coprod_w T^*_{X_w} X$. Furthermore, if $p : T^*X \longrightarrow X$ denotes the projection, then one has:

$$H^*(\mathcal{L} \overset{!}{\otimes} \mathcal{L}) = H^*(i^! \, Sp_\Delta(\mathcal{L} \boxtimes \mathcal{L})) = H^*(p_* F) = H^*(F)$$

5. What's next?

Conjecture 5.1. There exists a contravariant (anti)-involution $\varkappa$ on $\mathcal{D}^b O_{mixed}$, the bounded derived category, taking M_y to $M_{w_0 \cdot y}$ and interchanging simple and projective modules as follows: $L_y \overset{\varkappa}{\rightleftarrows} P_{w_0 \cdot y}$.

5.2. Being a functor on a triangulated category, the involution of 5.1 must, of course, commute with the translation functor $[\cdot]$. It should not commute, however, with the Tate twist $(\cdot)$. One should have instead: $\varkappa(M(i)) = \varkappa(M)(i)[i]$.

5.3. The conjecture would yield an algebra isomorphism: $A_0 \xrightarrow{\sim} A^!_0$.

The existence of such an isomorphism is suggested by 4.11.

5.4. The conjecture would give an explanation to the following known results: (i) BGG-reciprocity: $(P_y : M_w) = (M_w : L_y)$; (ii) the above mentioned inversion formula for the Kazhdan-Lusztig polynomials.

5.5. Choose a Cartan subalgebra $\underline{h} \subset \underline{b}$. Let $\underline{b}^-$ be the Borel subalgebra of $\underline{g}$, opposite to $\underline{b}$, and let O^-_{mixed} be the O-category defined with respect to $\underline{b}^-$. It is better, perhaps, to restate the conjecture, as giving an equivalence: $\mathcal{D}^b O_{mixed} \xrightarrow{\sim} \mathcal{D}^b O^-_{mixed}$, taking M_y to M^-_y and L_y and P^-_y. Such an equivalence is compatible with an isomorphism: $\mathrm{Ext}^*(M_y, M_w) \cong \mathrm{Ext}^*(M^-_w, M^-_y)$, $(y \leqslant w)$. Both sides here are equal to $H^*_c(B \cdot w \cap B^- \cdot y)$, where w, y are $\underline{h}$-fixed points in X and $B \cdot w$, $B^- \cdot y$ denote the orbits of the opposite Borel subgroups, corresponding to $\underline{b}$ and $\underline{b}^-$.

5.6. (after conversation with B.Feigin). There should be a similar equivalence: $\mathcal{D}^b O \xrightarrow{\sim} \mathcal{D}^b O^-$ for derived O-categories over Kac-Moody algebras. It should exchange modules with central charges c and $(26 - c)$ and it seems to be connected with Feigin's duality (defined using the Feigin-Fuks semi-infinite cohomology).

5.7. Let T be a maximal torus of G with Lie algebra $\underline{h} \subset \underline{b}$. Any perverse sheaf $M \in O$ is, clearly, a T-monodromic perverse sheaf, relative to the natural T-action on X. The monodromy action of $\pi_1(T)$ on M turns out to be unipotent, giving rise to a nilpotent $\underline{h}$-action on M. The same action can be described in terms of $U(\underline{g})$-modules as a nilpotent part of the natural $\underline{g}$-action, restricted to $\underline{h}$. That is, let x denote an operator on a $\underline{g}$-module $M \in O$, given by an element of $\underline{h}$. The point is, that the nilpotent part of x commutes with the $U(\underline{g})$-action. Thus, we get a homomorphism: $S(\underline{h}) \longrightarrow \mathrm{Hom}_O(M, M)$. Its image belongs to the center of $\mathrm{Hom}_O(M, M)$, since a monodromy action commutes with Hom's.

Let J denote the ideal of $S(\underline{h})$ generated by W-invariants: $S(\underline{h})^W_+$. The homomorphism: $S(\underline{h}) \to \mathrm{Hom}(M,M)$ is trivial on J. It is likely, in particular, that the resulting map: $S(\underline{h})/J \longrightarrow \mathrm{Hom}_O(P_e, P_e)$, ($e \in W$, the identity) is an algebra isomorphism. Anyway, we obtain an algebra homomorphism: $S(\underline{h})/J \longrightarrow A_O = \mathrm{Hom}_O(P, P)$.

5.7. The natural morphism in $\mathcal{D}^b(X)$: $\mathbb{C}_X \longrightarrow \underline{R\ \mathrm{Hom}}(\mathcal{L}, \mathcal{L})$, ($\mathcal{L} = \oplus \mathcal{L}_w$) gives rise to an algebra homomorphism: $H^*(X) \longrightarrow A_O^! = \mathrm{Ext}^*(\mathcal{L}, \mathcal{L})$. It is likely to correspond to the above homomorphism: $S(\underline{h})/J \longrightarrow A_O$ via our duality 5.2.

5.8. Suppose that $y \leqslant w$. Is it true that $\mathrm{Hom}_O(P_y, P_w) \neq 0$ iff $y \underset{LR}{\leqslant} w$ in the sense of Kazhdan-Lusztig (cf. 4.7)?

5.9. Our involution 5.1 seems to be related to the order reversing involution on the set of primitive ideals, defined by G. Lusztig.

5.10. It seems likely that the category of "mixed Harish-Chandra modules" over a real reductive group is formal.

5.11. Let us identify the categories O and $\mathrm{Mod}A_O$. The Verdier duality on O then gives rise to a grading preserving anti-involution $* : A_O \longrightarrow A_O$. The dual of a A_O-module M is defined to be the right A_O-module $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ endowed with a left A_O-module structure via the anti-involution on A_O.

The anti-involution on A_O induces a similar one on $A_O^!$. In the presentation : $A_O = TV/\langle W\rangle$ (see p.1) the involution on $A_O^! = TV^*/\langle W^\perp\rangle$ is induced by the map : $V^* \longrightarrow V^*$, adjoint to $* : V \longrightarrow V$. There is an alternative description, based on the presentation : $A_O^! = \oplus_{y,w} \mathrm{Ext}^*(L_y, L_w)$ and on self-duality $L_w^* = L_w$ of simple modules. The anti-involution on $A_O^!$ then arises as a composite :

$$\mathrm{Ext}^*(L_y, L_w) \xrightarrow{\sim} \mathrm{Ext}^*(L_w^*, L_y^*) \xrightarrow{\sim} \mathrm{Ext}^*(L_w, L_y)$$

The isomorphism : $A_O \xrightarrow{\sim} A_O^!$ of 5.3 is expected to commute with the anti-involutions on A_O and $A_O^!$.

Let $e_w = \mathrm{id}_{P_w} \in A_O$ denote the idempotent (of degree 0) associated with the projective $P_w \in O$. Then $A_O \cdot e_w$ is a projective A_O-module and $\mathbb{C} \cdot e_w = A_O \cdot e_w / A_O^+ \cdot e_w$ is its simple quotient, corresponding to L_w. The self-duality of L_w implies that: $e_w^* = e_w$. Applying $*$ to the vector space $e_w \cdot A_O \cdot e_y \cong \mathrm{Hom}(P_w, P_y)$ we get

<u>Corollary 5.11.</u> $\mathrm{Hom}(P_w, P_y) = e_w \cdot A_O \cdot e_y = e_y \cdot A_O \cdot e_w = \mathrm{Hom}(P_y, P_w)$.

In particular, the Cartan matrix $\dim \mathrm{Hom}(P_w, P_y)$ is symmetric (cf. [BGG] , [MiV]).

5.12. Let Vect denote the category of finite-dimensional vector spaces. It can be shown easily that any exact functor $F : O \longrightarrow$ Vect is representable, i.e. of the form: $M \longmapsto \mathrm{Hom}_O (P_F, M)$, where P_F is a projective module determined by F. Hence F is isomorphic to a direct sum of indecomposable functors $F_w : M \longmapsto \mathrm{Hom}_O (P_w, M)$.

Given a Bruhat cell X_w, we choose a point $x \in X_w$ and a holomorphic function f, defined on a neighborhood of x, such that the covector df_x is a generic point of $T^*_{X_w} X$. Let $\phi_w : O \longrightarrow$ Vect be the functor, assigning to $M \in O$ the stalk at x of the vanishing cycles sheaf $\phi_f(M)$. This is known to be an exact functor, commuting with Verdier duality. Furthermore, one can check that : $\phi_w = F_w$ + a sum of F_y's with $y > w$. The multiplicity matrix $(\phi_w : F_y)$ is therefore unipotent, so that F_w's can be expressed in terms of ϕ_w's.

It can be expected, following results of Beilinson, Deligne, MacPherson, Verdier and others on the extension problem for perverse sheaves, that there is a combinatorial description of the category of perverse sheaves on a stratified space. To be more specific, let $X = \bigsqcup X_w$ be the Bruhat cell stratification of X, the Flag manifold. It is expected that the algebra $\oplus_{y,w} \mathrm{Hom}(\phi_y, \phi_w)$ can be described geometrically and that it is generated, for instance, by elements $a_{y,w} \in \mathrm{Hom}(\phi_y, \phi_w)$, one for each couple (y, w) such that

$$\dim (\overline{T^*_{X_y} X} \cap \overline{T^*_{X_w} X}) = \dim T^*_{X_y} X - 1 \qquad (5.12.1)$$

Expressing now ϕ's in terms of P_y's and using 4.7, we see that $\mathrm{Hom}(\phi_y, \phi_w) \neq 0$, provided $\mu(y, w) \neq 0$. Something seems to be wrong here, however, for it is known that $\mu(y, w)$ might happen to be non-trivial even when (5.12.1.) fails.

5.13. Let $\hat{O}$ denote the category of finitely-generated $U(\underline{g})$-modules M with the properties: (i) there exists an integer $r = r(M) >> 0$ such that $Z_+^r \cdot M = 0$; (ii) the $U(\underline{b})$-action on M is locally-finite. Clearly $O \subset \hat{O}$ and both categories have the same simple objects L_w.

Let G denote a simply-connected Lie group associated to $\underline{g}$ and let $B = T \cdot U$ be the Borel subgroup, corresponding to $\underline{b}$. The projection: $\hat{X} = G/U \longrightarrow X = G/B$ has a natural structure of a principal T-bundle, relative to a T-action on X "on the right". The category $\hat{O}$ is known (Beilinson-Bernstein) to be equivalent to the category of regular monodromic modules on $\hat{X}$ with unipotent monodromy, that are smooth along B-orbits on $\hat{X}$.

Next, we define $\hat{O}_{mixed}$ to be a similar category of monodromic geometric mixed Hodge modules on $\hat{X}$. Let $\hat{P}_w$ denote an indecomposable projective cover of L_w in $\hat{O}_{mixed}$ ($\hat{P}_w$ is a pro-object, i.e. a projective limit of objects of $\hat{O}_{mixed}$). Set $\hat{P} = \oplus_w \hat{P}_w$ and let $A_{\hat{O}} = \oplus_{i \in Z} \mathrm{Hom}(\hat{P}, \hat{P}(i))$. The graded algebra $A_{\hat{O}}$ has a natural $S(\underline{h})$-algebra structure, arising from the monodromy-action along the fibres of $\hat{X} \longrightarrow X$ (this is <u>not</u> the action, considered in n. 5.7).

We turn now to an analog of 5.1 for $\hat{O}_{mixed}$. One can not expect to have an isomorphism $A_{\hat{O}} \cong A^!_{\hat{O}}$ because $\dim_{\mathbb{C}} A_{\hat{O}} = \infty$, while $\dim A^!_{\hat{O}} = \dim \mathrm{Ext}^*_{\hat{O}} (L, \oplus_i L(i)) < \infty$. To state a correct version, we introduce the torus T^*, dual to T, a reductive group G^*, dual to G in the sense of Langlands and having T^* as a maximal torus, a Borel subgroup B^*, containing T^*, and the Flag manifold $X^* = G^*/B^*$. Let $\mathcal{L}^*_w$, $(w \in W)$ denote the Intersection cohomology complex, corresponding to the B^*-orbit $X^*_w \subset X^*$, and let $\mathcal{L}^* = \oplus_{w \in W} \mathcal{L}^*_w$. We view $\mathcal{L}^*$ as a T^*-equivariant perverse sheaf on X^*.

Let BT^* denote the universal classifying space for the torus T^* and $\mathrm{Ext}_{T^*}(\mathcal{L}^*, \mathcal{L}^*)$, the T^*-equivariant Ext-group. The latter has a natural $H^*(BT^*)$-algebra structure. We recall that $H^*(BT^*) \cong S(\underline{h})$.

<u>Conjecture 5.13.</u> There is a graded algebra isomorphism:

$$\oplus_{i \in Z} \mathrm{Hom}_{\hat{O}_{mixed}} (\hat{P}, \hat{P}(i)) \cong \mathrm{Ext}_{T^*} (\mathcal{L}^*, \mathcal{L}^*)$$

The $S(\underline{h})$-module structure on the left-hand side should correspond, via the conjecture, to the $H^*(BT^*)$-module structure on the right-hand side.

For a general construction of equivariant Ext-groups the reader is referred to [Gi]. In the special case, we need, these groups can be defined directly as follows. Let : $ET^* \longrightarrow BT^*$ denote the standard model for a principal universal T^*-bundle with BT^* being a product of $\mathbb{CP}^\infty$'s. Set $XT^* := ET^* \times_{T^*} X^*$ and $XT^*_w = ET^* \times_{T^*} X^*_w$, a subvariety of XT^*. Let $\mathcal{L}_{T^*}$ denote the direct sum of the Intersection cohomology complexes on all XT^*_w's. This is a perverse sheaf on XT^*. We set, by definition: $Ext_{T^*}(\mathcal{L}^*, \mathcal{L}^*) = Ext(\mathcal{L}_{T^*}, \mathcal{L}_{T^*})$.

We have much benefited from conversations with a number of people. We are especially grateful to B.Feigin, S.Gel'fand, M.Končevich, S.Khoroshkin and V.Shehtmann.

References

[Be] A.Beilinson. On the derived category of perverse sheaves. - Preprint. Moscow, 1985.

[BGG] J.Bernstein, I.M.Gel'fand, S.Gel'fand. A category of g-modules. - Funct. Anal. Appl. 10 (1976), 87-92.

[GaJ] O.Gabber, A.Joseph. Towards the Kazhdan-Lusztig conjecture. - Ann. Sci. E.N.S. 14 (1981), 261-302.

[Gi] V.Ginsburg. Kähler geometry and equivariant cohomology. - Funct. Anal. Appl. 21 (1987) N 1 (to appear).

[KL] D.Kazhdan, G.Lusztig. Representations of Coxeter groups and Hecke algebras. - Invent. Math. 53 (1979), 165-184.

[Kap] M.Kapranov. The derived category of coherent sheaves on quadrics. - Func. Anal. Appl. 20 (1986), N 2.

[MacV] R.MacPherson, K.Vilonen. Elementary construction of perverse sheaves. Invent. Math. 84 (1986), 403-425.

[MiV] R.Mirollo, K.Vilonen. BGG reciprocity on Perverse sheaves. Preprint, 1986.

[Priddy] S.Priddy. Koszul algebras.-Transactions of AMS. 152 (1970), 39-60.

[Ba] D. Barbasch. On the Jantzen filtration. - Ann. Sci. E.N.S. 16 (1983), 489-494.