Sep 24, 2020

10) Category $\mathcal{O}$ has enough projectives:

i.e, $\forall M \in \mathcal{O}$ $\exists$ proj $P \in \mathcal{O}$ and epimorphism: $P \to M$

$$\{\text{Indecomposable projs in } \mathcal{O}\}/_{\cong} = \{P(\lambda) \mid \lambda \in \mathfrak{h}^*\}$$

$P(\lambda)$: proj cover of $L(\lambda)$ $(M(\lambda))$ $\quad P(\lambda) \to M(\lambda) \to L(\lambda)$

<u>Thm</u> $\quad \lambda \in \mathfrak{h}^*$

1) $M(\lambda) = P(\lambda) \iff \lambda$ is dominant

($\Rightarrow$ maximal in $W_{[\lambda]}.\lambda$)

2) $M(\lambda) = L(\lambda) \iff \lambda$ is antidominant

<u>Sketch of pf</u>

1) "$\Leftarrow$"

$$\begin{tikzcd} & M(\lambda) \arrow[dl, dashed] \arrow[d] & v_+ \arrow[d] \\ M \arrow[r, two heads] & N & n \\ & m \arrow[ur] & \end{tikzcd}$$

can assume $M, N \in \mathcal{O}_\lambda$

m is a maximal vector b/c λ is dom.

"$\Rightarrow$" BGG reciprocity

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)] = \delta_{\lambda\mu}$$

if λ not dom $\quad \exists\, \alpha$ s.t $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{<0}$

$\Rightarrow s_\alpha . \lambda = s_\alpha(\lambda+\rho) - \rho = \lambda + \rho - \langle \lambda+\rho, \alpha^\vee \rangle \alpha - \rho$

$= \lambda - \langle \lambda+\rho, \alpha^\vee \rangle \alpha \quad > \lambda$

$[M(s_\alpha . \lambda) : L(\lambda)] \neq 0$

(**Thm** Let $\lambda, \mu \in \mathfrak{h}^*$

(a) (Verma) If μ is strongly linked to λ then $M(\mu) \hookrightarrow M(\lambda)$, in particular

$$[M(\lambda) : L(\mu)] \neq 0$$

(b) (BGG) If $[M(\lambda) : L(\mu)] \neq 0$, then μ is strongly linked to λ.

[$\mu \uparrow \lambda$ if $\mu = \lambda$ or $\exists\, \alpha > 0$ s.t. $\mu = s_\alpha . \lambda < \lambda$. More generally, μ is strongly linked to λ if

$$\mu = (s_{\alpha_1} \cdots s_{\alpha_r}) . \lambda \uparrow (s_{\alpha_2} \cdots s_{\alpha_r}) . \lambda \uparrow \cdots \uparrow s_{\alpha_r} . \lambda \uparrow \lambda \,]$$

15 (2) "$\Leftarrow$" $M(\lambda)$ has a unique simple submod

$L(\mu)$ $\mu \le \lambda$ $\mu = w.\lambda$ some $w \in W[\lambda]$.

"$\Rightarrow$" Thm (Verma)

$\lambda \in \mathfrak{h}^*$ $\alpha > 0$ $\mu := s_\alpha . \lambda \le \lambda$

Then $\exists$ embedding $M(\mu) \subset M(\lambda)$ (as in above)

$\square$

<u>Idea of enough projectives</u>

$\lambda \in \mathfrak{h}^*$ $\mu = \lambda + n\rho$ dominant $n \gg 0$

$\Rightarrow M(\mu)$ projective $\dim L(n\rho) < \infty \Leftarrow n\rho \in \Lambda^+$

$\Rightarrow P = M(\mu) \otimes L(n\rho)$ projective

$\Rightarrow P$ has a quotient $\cong M(\mu - n\rho) = M(\lambda)$

$\Rightarrow M(\lambda)$ & $L(\lambda)$ is the quotient of a proj mod.

Proceed by induction on length. $\square$

<u>Thm</u> Every projective in $\mathcal{O}$ has a standard filtration. i.e. subquotients are Verma mods

<u>Pf</u> $P(\lambda)$ direct summand of $M(\mu) \otimes L$ some μ dominant, $\dim L < \infty$

$M(\mu) \otimes L$ has std filtr. (with quotients iso to $M(\lambda+\mu)$, each occuring $\dim L_\mu$ times, μ ranges over wts of M)

$$(M(\mu) \otimes L \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes L)$$

$\square$

Intrinsic characterization of objects admitting std filtr.

For $M \in \mathcal{O}$, TFAE:

(a) M admits a standard filtration

(b) $\operatorname{Ext}^i_{\mathcal{O}}(M, M(\mu)^\vee) = 0 \quad \forall \mu \ \& \ i > 0$

(c) $\operatorname{Ext}^1_{\mathcal{O}}(M, M(\mu)^\vee) = 0 \quad \forall \mu$

<u>Thm</u> (BGG reciprocity)

Let $\lambda, \mu \in \mathfrak{h}^*$. Then

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)] = [M(\mu)^\vee : L(\lambda)]$$

<u>Pf</u> 1) If $M \in \mathcal{O}$ has standard filtration,

then $(M : M(\lambda)) = \dim \mathrm{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee)$

Argue by induction on filtr. length

$$0 \to N \to M \to M(\mu) \to 0$$

$$0 \to \mathrm{Hom}(M(\mu), M(\lambda)^\vee) \to \mathrm{Hom}(M, M(\lambda)^\vee) \to \mathrm{Hom}(N, M(\lambda)^\vee)$$

$$\to \overset{0}{\cancel{\mathrm{Ext}^1(M(\mu), M(\lambda)^\vee)}} \to \cdots$$

2) $M \in \mathcal{O}$ $\dim \mathrm{Hom}(P(\lambda), M) = [M : L(\lambda)]$

by induction on length of M

both side additive in M $\square$

$$(P(\lambda) : M(\mu)) = \dim \mathrm{Hom}(P(\lambda), M(\mu)^\vee) = [M(\mu)^\vee : L(\lambda)]$$

Translation functors

$$T_\lambda^\mu : \mathcal{O} \to \mathcal{O}$$
$$M \mapsto pr_\mu (L(\bar{\nu}) \otimes (pr_\lambda M))$$

$\mu - \lambda \in \Lambda$ $\{\bar{\nu}\} = \Lambda^+ \cap W(\mu-\lambda)$ (λ, μ compatible)

$pr_\lambda : \mathcal{O} \to \mathcal{O}_{\chi_\lambda}$

- T_λ^μ is exact, commutes with duality functor, takes projectives to projectives.
- T_λ^μ is left and right adjoint to T_μ^λ
- λ, μ dominant integral

  $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ $T_\mu^\lambda : \mathcal{O}_\mu \to \mathcal{O}_\lambda$

  mutually quasi-inverse equivalences
- μ dominant integral

  $T_{-\rho}^\mu (M(-\rho)) \cong P(w_0 . \mu)$

3) $P(\lambda) \cong P(\lambda)^\vee \Leftrightarrow \lambda$ is antidominant

λ antidominant. Any standard filtr. of $P(\lambda)$ involves each $M(w\cdot\lambda)$ with $w \in W_{[\lambda]}$ precisely once as a quotient

BGG reciprocity $\Rightarrow [M(w\cdot\lambda) : L(\lambda)] = 1 \quad \forall w \in W_{[\lambda]}$

Remark on exts: e.g. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$

N : 2-diml $U(\mathfrak{b})$-mod $\quad eN = 0 \quad h : \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$0 \to M(\lambda) \to M = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N \to M(\lambda) \to 0 \quad M \notin \mathcal{O}$

$(\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\lambda)) = 0$, more generally, M h.w.m of h.w μ, $\lambda \nless \mu$ $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M) = 0$

Any s.e.s $0 \to M \xrightarrow{f} E \xrightarrow{g} M(\lambda) \to 0$ in $\mathcal{O}$ splits)

Principal block

$$\mathcal{O}_0 = \mathcal{O}_{\chi_0} \qquad \mathrm{Irr}\,\mathcal{O}_0 \overset{\sim}{\longleftrightarrow} W$$

$$L(w.0) \leftrightarrow w \qquad 0 \text{ regular dom.}$$

$w_0 . 0 = -2\rho$ regular antidom.

Example $\mathfrak{sl}_2$

$\mathcal{O}_0$: five indecomposable modules

$$P(0) = M(0), \quad L(0), \quad M(0)^\vee$$

$$M(-2) = L(-2) \cong M(-2)^\vee$$

$$P(-2) \cong M(-1) \otimes L(1) \cong P(-2)^\vee$$

$$0 \to M(0) \to P(-2) \to M(-2) \to 0$$

$$0 \to M(-2) \to P(-2) \to M(0)^\vee \to 0$$

$$0 \to M(-2) \to M(0) \to L(0) \to 0$$

$$0 \to L(0) \to M(0)^\vee \to M(-2) \to 0$$

Most $\mathcal{O}_0$'s are of wild type (inf. many indecomps can not be para. in a resonable way)

Exercise $z \in Z(\mathfrak{g})$ $\quad z|_{P(-2)} \neq 0 \quad z^2|_{P(-2)} = 0.$

Brylinski – Kashiwara / Beilinson – Bernstein

$\mathcal{O}_0'$: consists of M

1) f.g $U(\mathfrak{g})$-mod
2) locally $U(\mathfrak{b})$-finite
3) $IM = 0$ $\quad I = \langle Z(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{g} \rangle$

i.e, $Z(\mathfrak{g})$ acts with trivial central char.

$\mathfrak{h}$-action not necessarily s.s.

(closed under submod, quotient, not closed under extension)

$X = G/B$

localisation (c.f Peter, Dougal)

$$\mathcal{O}_0' \cong \mathcal{M} \overset{RH}{\cong} P_N(G/B)$$

$\mathcal{M} = \{$r.h. D_X-modules whose char varieties $\subset \bigcup_{w \in W} T^*_{X_w} X \}$ $\quad X = G/B$, $\quad X_w = BwB/B$

$(P_{(B)}(G/B))$

Soergel $\mathcal{O}_0' \cong \mathcal{O}_0$, $\quad P_N(G/B) \cong P_B(G/N)$

$$\mathcal{O}_0 \cong P_B(G/N)$$

(monodromic D-modules /sheaves)

$\mathcal{O}_0' \cap \mathcal{O}_0 \cong P_B(G/B)$

$M(w.0) \quad L(w.0) \quad M(w.0)^\vee \in \mathcal{O}_0' \ \& \ \mathcal{O}_0$

$$\mathcal{O}_0 \longrightarrow P_N(G/B)$$

$$M(w.0) \longmapsto j_! \mathbb{C}_{X_{ww_0}}[-]$$

$$M(w.0)^\vee \longmapsto j_* \mathbb{C}_{X_{ww_0}}[-]$$

$$L(w.0) \longmapsto IC_{X_{ww_0}}[-]$$

In particular

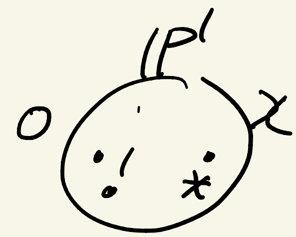
$L(0) =$ trivial module $\Leftrightarrow IC_{X_{w_0}}$

$M(w_0.0) \cong L(w_0.0) \cong M(w_0.0)^\vee \Leftrightarrow$ skyscraper

big proj $\quad P(w_0.0) \Leftrightarrow$?

$\underline{\mathfrak{sl}_2}$ What does $P(-2)$ correspond to?

$$\mathcal{D}_X / \langle z\partial z \rangle$$



Nearby cycles.

Koszul self-duality of $\mathcal{O}_0$

Let $P = \bigoplus_{w \in W} P(w.0)$ (projective generator of $\mathcal{O}_0$, i.e, proj. obj surjecting to each simple)

Morita theory

$$\Rightarrow \mathrm{Hom}_{\mathcal{O}}(P,-): \quad \mathcal{O}_0 \xrightarrow{\cong} \text{mod}^{f.g}-\mathrm{End}_{\mathcal{O}}(P)$$

Let $L = \bigoplus_{w \in W} L(w.0)$

<u>Thm</u> (Soergel '90)

There exists an isomorphism of finite dimensional $\mathbb{C}$-algebras

$$\mathrm{End}_{\mathcal{O}}(P) \cong \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L,L) = \bigoplus_{i \geq 0} \mathrm{Ext}^{i}_{\mathcal{O}}(L,L)$$

(ring: cup product)

Moreover, $\mathrm{Ext}^{\bullet}_{\mathcal{O}}(L,L)$ is a <u>koszul ring</u>.

(noncanonical isom.)

$A = \bigoplus_{j \geq 0} A_j$: 1) A_0 semisimple

2) A_0 (as a graded left A-mod) admits a graded proj. res $\cdots \to P^2 \to \cdots \to P^0 \twoheadrightarrow A_0$ s.t $P^i = AP^i_i$

Self-duality

$A = \bigoplus_{i \geq 0} A_i$ left finite koszul ring (A_1 is f.g as left A_0-mod)

$\Rightarrow$ $E(A) := \mathrm{Ext}^{\bullet}_{A}(A_0, A_0)$ is also left finite koszul ring & $E(E(A)) = A$.

($E(A)$ is formal)

Let $A = \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L, L) \cong \mathrm{End}_{\mathcal{O}}(P)$

under $\mathrm{mod}^{f.g}\text{-}A \xrightarrow{\cong} \mathcal{O}_0$

$A_0 \mapsto L$

$\Rightarrow \mathrm{Ext}^{\bullet}_{A}(A_0, A_0) \cong \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L, L)$

i.e. $E(A) \cong A$

(Bezrukavnikov's lemma $\Rightarrow$ A is koszul)

[BGS]: "Correct proof": using mixed geometry.

Idea of proof of Soergel's theorem

Let $R = S(\mathfrak{h}^*)$ & $C = R/\langle R_+^W \rangle$

1) $\mathrm{End}_{\mathcal{O}}(P(w_0 . 0)) \cong C$ $\quad \left(Z(\mathfrak{g}) \twoheadrightarrow \mathrm{End}_{\mathcal{O}}(P(w_0.0)) \right)$

2) Soergel's $\mathbb{V}$-functor

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}}(P(w_0.0), -) : \mathcal{O}_0 \to C\text{-mod}$$

3) $\mathbb{V}|_{\mathrm{Proj}(\mathcal{O}_0)}$ is fully faithful

(Another proof via geometry: [BBM] Tilting exercises

$$\mathbb{V} \cong \pi : \mathcal{A} = \mathrm{Perv}_{(B)}(G/B) \to \mathcal{A}_0 = \mathcal{A}/\mathcal{A}_{>0}$$

$\mathbb{V}(L(w.0))$

$= \begin{cases} \mathbb{C} & w = w_0 \\ 0 & \text{otherwise} \end{cases}$

$\mathcal{A}_{>0}$: Serre subcat generated by IC_{X_w} $w \neq 1$

Radon transf: $R^!_{w_0} = pr_{2!}\, pr_1^* [-]$

$$X^2_{w_0} = G(X_1 \times X_w)$$

$$\begin{tikzcd} & X^2_{w_0} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & X \end{tikzcd}$$

$\mathbb{V}(M(w.0)) \cong \mathbb{C}$ $\forall w$

$R^{!}_{w_0}(T_w) = P_{ww_0}$ $\quad \mathbb{V}\circ R^{!}_{w_0} \cong \mathbb{V}$

$\uparrow$ tilting $\qquad \mathbb{V}|_{\text{tilting}}$ fully faithful)

M tilting : $\forall w$, $j_w^! M$, $j_w^* M$ are perverse sheaves on X_w $\quad j_w : X_w \hookrightarrow X$

$\operatorname{Supp} T_w = \overline{X_w}$ $\quad j_w^! T_w = \underline{\mathbb{C}}[\,]$

$\mathbb{V}|_T$ fully faithful

Lemma The socle of M_w $(M(w.0))$ / cosocle of $M_w^\vee$ is $\mathcal{L}_e$ $(L(w_0.0))$

$\mathcal{A}$ abelian cat $\quad \mathcal{B} \subset \mathcal{A}$ Serre subcat

$${}^\perp\mathcal{B} := \{A \in \mathcal{A} : \operatorname{Hom}(A, X) = 0 \quad \forall X \in \mathcal{B}\}$$

$$\mathcal{B}^\perp := \{A \in \mathcal{A} : \operatorname{Hom}(X, A) = 0 \quad \forall X \in \mathcal{B}\}$$

$\Rightarrow$ if $A \in {}^\perp\mathcal{B}$ $\quad B \in \mathcal{B}^\perp$, then

$$\operatorname{Hom}_{\mathcal{A}}(A, B) \simeq \operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(A, B)$$

Lemma $\Rightarrow$ $M_w \in \mathcal{A}_{>0}^{\perp}$ $\quad M_w^{\vee} \in {}^{\perp}\mathcal{A}_{>0}$

$\Rightarrow$ $T \subset {}^{\perp}\mathcal{A}_{>0} \cap \mathcal{A}_{>0}^{\perp}$ $\qquad \square$

4) $$\mathbb{IH}^{\bullet} : \mathrm{SemiSimple}_{(\check{B})}(\check{G}/\check{B}) \to \underset{\substack{\rotatebox{90}{$\cong$} \\ C}}{H^{\bullet}(\check{G}/\check{B})}\text{-mod}$$

is fully faithful

$(\mathrm{End}^{\bullet}_{D(\check{X})}(\underline{\mathbb{C}}_{\check{X}}) \circlearrowright \mathrm{Hom}^{\bullet}_{D(\check{X})}(\underline{\mathbb{C}}_{\check{X}}, \mathcal{F}) = \mathbb{IH}^{\bullet}(\mathcal{F}))$ (c.f. Gufang's talk)

5)
$$\begin{tikzcd}
& \mathrm{SemiSimple}_{(\check{B})}(\check{G}/\check{B}) \arrow[r, "h"] \arrow[d, "\mathbb{IH}^{\bullet}"] \arrow[dr, phantom, "\circlearrowright \text{ Commutes}"] & \mathcal{H}_W \arrow[d, "q=1"] \\
\mathrm{Proj}(\mathcal{O}_0) \arrow[r, "\mathbb{V}"] & C\text{-mod} \arrow[r, "b"] & W
\end{tikzcd}$$

$\mathbb{V}(P(w.0)) \cong \mathbb{IH}^{\bullet}(IC_{\check{X}_w})$ $\qquad \mathbb{V}(P) \mapsto \sum (P : M(x.0))\, x$

$\Downarrow$

Kazhdan-Lusztig conjecture.

$\pi_s : G/B \to G/P_s \qquad s \in S$

$$\mathbb{IH}^{\bullet}(\pi_s^{*} \pi_{s*} \mathcal{F}) \cong C \otimes_{C^s} \mathbb{IH}^{\bullet}(\mathcal{F})$$

$$\mathbb{V} \circ \theta_s(-) \cong C \otimes_{C^s} \mathbb{V}(-)$$

$w = s_1 \cdots s_r$ reduced expression

$$\pi_{s_r}^* \pi_{s_r *} \cdots \pi_{s_1}^* \pi_{s_1 *} IC_{X_1}$$ $1 =$ identity in W

$$\cong IC_{X_w}[-] \oplus \text{lower terms}$$

$$\Theta_{s_r} \circ \cdots \circ \Theta_{s_1}(P(0))$$

$$\simeq P(w.0) \oplus \bigoplus_{y<w} P(y.0)^{\oplus m_y}$$

$\mathbb{V}(P(0)) \cong \mathbb{H}^\bullet(IC_{X_1}) \cong \mathbb{C}$ (trivial C-mod: Image of R^+ acts by 0)

$$h(\pi_s^* \pi_{s*} \mathcal{F}) = h(\mathcal{F})(T_s + 1)$$

$$b(\mathbb{V}(\Theta_s P)) = b(\mathbb{V}(P))(s+1)$$

Θ_s: Wall-crossing functors choose μ integral s.t

$\mu \in$ closure ρ-shifted dominant Weyl chamber

$\mu \in H_{\alpha_s}$ $\mu \notin H_\alpha$ $\alpha \neq \alpha_s$ $(\mathrm{stab}_{(W,\cdot)}(\mu) = \{1, s\})$

define $\Theta_s = T_\mu^0 \circ T_0^\mu$

$0 \to M_w \to \Theta_s M_w \to M_{ws} \to 0$ if $ws > w$ $M_w = M(w.0)$

$0 \to M_{ws} \to \Theta_s M_w \to M_w \to 0$ if $ws < w$

<u>Example</u> $\mathfrak{sl}_2$

$$\mathcal{O}_0 \simeq \{\text{representations of quiver } \bullet \underset{f}{\overset{e}{\rightleftarrows}} \bullet \text{ s.t. } ef = 0\}$$

$$M \longmapsto \begin{tikzcd} M_0 \arrow[d, shift left, "f"] \\ M_{-2} \arrow[u, shift left, "e"] \end{tikzcd}$$

$M(0)$

$$\begin{tikzcd} \cdot\, v_0 \arrow[d, bend left, "f"] \\ \cdot\, v_1 \arrow[u, bend left, "e"] \end{tikzcd} \qquad \begin{array}{l} ev_1 = 0 \\ fv_0 = v_1 \end{array}$$

$M(0)^\vee$

$$\begin{tikzcd} v_0 \arrow[d, bend left, "f"] \\ v_1 \arrow[u, bend left, "e"] \end{tikzcd} \qquad \begin{array}{l} ev_1 = v_0 \\ fv_0 = 0 \end{array}$$

$L(0)$

$$\begin{tikzcd} \cdot\, v_0 \arrow[d, bend left, "f"] \\ \cdot\, 0 \arrow[u, bend left, "e"] \end{tikzcd} \qquad e$$

$M(-2)$ $= L(-2)$

$$\begin{tikzcd} \circ\, 0 \arrow[d, bend left, "f"] \\ \circ\, v_1 \arrow[u, bend left, "e"] \end{tikzcd}$$

$P(-2)$

$$\begin{tikzcd} \cdot\, v_0 \arrow[d, bend left, "f"] \\ \cdot\, v_1, v_2 \arrow[u, bend left, "e"] \end{tikzcd} \qquad \begin{array}{ll} ev_1 = v_0 & fv_0 = v_2 \\ ev_2 = 0 & \end{array}$$

$$\dim \mathrm{Hom}_{\mathcal{O}}(P(0), P(0)) = \dim \mathrm{Hom}_{\mathcal{O}}(P(0), P(-2))$$

$$= \dim \mathrm{Hom}_{\mathcal{O}}(P(-2), P(0)) = 1$$

$$\dim \mathrm{End}_{\mathcal{O}}(P(-2)) = 2$$

$$\mathrm{End}_{\mathcal{O}}(P(-2)) \cong \mathbb{C}[x]/x^2 \cong C \qquad \left(\begin{array}{l}\text{Recall } z \in Z(\mathfrak{g}) \\ \qquad z^2 = 0\end{array}\right)$$

$$\dim \mathrm{Hom}(L(0), L(0)) = \dim \mathrm{Hom}(L(-2), L(-2)) = 1$$

$$\dim \mathrm{Ext}^1(L(0), L(-2)) = \dim \mathrm{Ext}^1(L(-2), L(0)) = 1$$

$$\dim \mathrm{Ext}^2(L(0), L(0)) = 1$$

Koszul duality

Thm A : Koszul ring, 1) A f.g as A_0-mod (both left & right) 2) $E(A)$ right noetherian.

Let $A^! = E(A)^{opp}$

There is an equivalence of triangulated categories

(Koszul duality functor) $K : D^b(A\text{-gmod}^{f.g}) \longrightarrow D^b(A^!\text{-gmod}^{f.g})$

s.t $K(M\langle n\rangle) = (KM)[-n]\langle n\rangle$ canonically.

etc

K : simple $\to$ proj   inj $\to$ simple

Let $\mathcal{O}_0^{mix} = A\text{-gmod}^{fg}$

<u>Corollary</u> There exists an equivalence of triangulated categories over $\mathbb{C}$

$$K : D^b(\mathcal{O}_0^{mix}) \to D^b(\mathcal{O}_0^{mix})$$

and there exist lifts of simple, standard, projective modules s.t.

$$\widetilde{L(x.0)} \longmapsto \widetilde{P(xw_0^{-1}.0)} \qquad (A = A^{op})$$

$$\widetilde{M(x.0)}^\vee \longmapsto \widetilde{M(xw_0^{-1}.0)} \quad \text{etc}$$

$\Rightarrow$ Explain "strange formula"

$$[M(x.0), L(y.0)] = \sum \dim \operatorname{Ext}^i (M(w_0x.0), L(w_0y.0))$$