BGG category $\mathcal{O}$ Sep 17, 2020

<u>References</u> [Humphreys 2008] [Gaitsgory 2005] [BGS 1996] [EMTW 2020] [Soergel 90] ....

$\mathfrak{g}$ - semisimple complex Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ $\quad \Phi^+ \leftrightarrow \mathfrak{n}$

[Bernstein - Gelfand - Gelfand Russian "basic" 1970s]

<u>Def</u> The BGG category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-modules consisting of modules M s.t

(1) finitely generated $U(\mathfrak{g})$-mod

(2) $\mathfrak{h}$-semisimple, i.e., $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$

(3) locally $\mathfrak{n}$-finite:

$\forall v \in M \qquad \dim U(\mathfrak{n}) v < \infty$

Properties of cat $\mathcal{O}$:

1) all finite diml $U(\mathfrak{g})$-mods lie in $\mathcal{O}$

2) $\forall M \in \mathcal{O}$ $\dim M_\lambda < \infty$ $\forall \lambda \in \mathfrak{h}^*$

   wt of $M \subset \bigcup_{i=1}^{s} \{\lambda_i - \mathbb{N}\Phi^+\}$

3) $\mathcal{O}$ is a noetherian cat, i.e. each $M \in \mathcal{O}$ is a noetherian $U(\mathfrak{g})$-mod [$U(\mathfrak{g})$ Noetherian]

4) $\mathcal{O}$ is closed under submodules, quotients, finite direct sums (not closed under extensions)

5) $\mathcal{O}$ is an abelian cat

6) $M \in \mathcal{O} \Rightarrow M$ is $Z(\mathfrak{g})$-finite:

$\forall v \in M$ $\dim \operatorname{Span}\{z \cdot v \mid z \in Z(\mathfrak{g})\} < \infty$

7) $M \in \mathcal{O} \Rightarrow M$ f.g as $U(\mathfrak{n}^-)$-mod

Examples of modules in $\mathcal{O}$

Highest wt module of wt λ: $M = U(\mathfrak{g})v^+$

for some maximal vector $v^+ \in M_\lambda$.

wt vector & $\mathfrak{n}_+ v^+ = 0$

M has a unique max submodule (= sum of all proper submodules) and a unique simple quotient, M is indecomposable, $\dim \mathrm{End}_{\mathcal{O}} M = 1$.

Verma module (universal h.w.m)

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

$L(\lambda)$: unique simple quotient of $M(\lambda)$

More properties of $\mathcal{O}$

(8) $\{$irreducibles (up to iso) in $\mathcal{O}\} = \{L(\lambda) \mid \lambda \in \mathfrak{h}^*\}$

$$\dim \mathrm{Hom}_{\mathcal{O}}(L(\lambda), L(\mu)) = \delta_{\lambda\mu}$$

Central character

$\forall z \in Z(\mathfrak{g})$ z acts on a h.w.m of h.w $\lambda \in \mathfrak{h}^*$ by a scalar χ_λ

$\rightsquigarrow \chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ (central char asso. to λ)

Theorem (Harish-Chandra)

1) $\chi_\lambda = \chi_\mu \iff \lambda$ and μ are linked,

i.e $\lambda = w.\mu = w(\mu+\rho) - \rho$ $w \in W$

2) all central characters $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ are of the form χ_λ for some $\lambda \in \mathfrak{h}^*$

Harish-Chandra homomorphism

$$\gamma : Z(\mathfrak{g}) \xrightarrow{\cong} U(\mathfrak{h})^{(W,\cdot)} \cong \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}$$

$z = h + n$ $h \in U(\mathfrak{h})$ $n \in U(\mathfrak{g})\mathfrak{n}_+$

$\gamma(z) = h$ $\chi_\lambda(z) = \lambda(\gamma(z))$

Twisted Harish-chandra homomorphism

$$\psi : Z(\mathfrak{g}) \xrightarrow{\simeq} S(\mathfrak{h})^W \simeq \mathbb{C}[\mathfrak{h}^*]^W$$

$$\psi(z)(\lambda) = \gamma(z)(\lambda-\rho)$$

$$\Rightarrow \quad \chi_\lambda(z) = (\lambda+\rho)(\psi(z))$$

Example $\quad \mathfrak{sl}_2 \quad z = ef + fe + \frac{h^2}{2}$

$$\gamma(z) = \frac{h^2}{2} + h \quad \psi(z) = \frac{h^2-1}{2}.$$

Rmk It is better to think of $\mathfrak{h}$ as the universal Cartan. The maps γ, ψ does not depend on choice of $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ via $\mathfrak{h} \cong \mathfrak{b}/\mathfrak{n} \cong \mathfrak{t}$.

Note For $M \in \mathcal{O}$ $Z(\mathfrak{g})$ acts via generalised central character in general.

Let $\Lambda = \{ \mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta \}$

$\Lambda^+ = \{ \mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0} \ \forall \alpha \in \Delta \}$

<u>Thm</u> (finite diml modules)

1) $\dim L(\lambda) < \infty \iff \lambda \in \Lambda^+$

$\iff \dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu} \quad \forall \mu \in \mathfrak{h}^* \ w \in W$

2) (BGG resolution)

$\lambda \in \Lambda^+ \Rightarrow$ there is a resolution

$$M^\bullet \to L(\lambda) \qquad M^{-k} = \bigoplus_{\substack{w \in W \\ \ell(w) = k}} M(w.\lambda)$$

In particular, max submod $N(\lambda)$ of $M(\lambda)$

$$\cong \bigoplus_{\alpha_i \in \Delta} M(s_i.\lambda)$$

3) (Weyl char formula)

$$\lambda \in \Lambda^+ \Rightarrow \operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w.\lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w.0)}$$

Duality in $\mathcal{O}$

Fix $x_\alpha \in \mathfrak{g}_\alpha$ $y_\alpha \in \mathfrak{g}_{-\alpha}$ $\alpha \in \Phi^+$

s.t $\alpha \in \Delta$ $h_\alpha = [x_\alpha, y_\alpha]$ $\alpha(h_\alpha) = 2$

Anti-involution $\tau : \mathfrak{g} \to \mathfrak{g}$

$x_\alpha \mapsto y_\alpha$ $y_\alpha \mapsto x_\alpha$ $h_\alpha \mapsto h_\alpha$

[τ = transpose for $\mathfrak{sl}_n$, $-\tau$: Chevalley inv]

τ extends canonically to an anti-autom of $U(\mathfrak{g})$

M weight module $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$

Define $M^\vee = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*$ with twisted action of $U(\mathfrak{g})$

$$(x f)(v) = f(\tau(x) v) \qquad x \in U(\mathfrak{g})$$

(Note this is not the usual dual module

e.g. $\lambda \in \Lambda^+$ $L(\lambda)^* \cong L(-w_0\lambda)$ $(x.f)(v) = -f(x.v)$

$L(\lambda)^\vee \cong L(\lambda)$)

<u>Thm</u> 1) $M \in \mathcal{O} \Rightarrow M^\vee \in \mathcal{O}$

The duality functor $M \mapsto M^\vee$ induces a self-equivalence on cat $\mathcal{O}$: its square is naturally equivalent to the identity functor

2) $\lambda, \mu \in \mathfrak{h}^*$

a) $M(\lambda)^\vee$ has $L(\lambda)$ as its unique simple submodule, its other comp. factors $L(\mu)$: $\mu < \lambda$.

b) $\mathrm{Hom}(M(\lambda), M(\lambda)^\vee) \cong \mathbb{C}$

$(M(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow M(\lambda)^\vee)$

c) $\mathrm{Hom}(M(\lambda), M(\mu)^\vee) = 0 \quad \lambda \neq \mu$

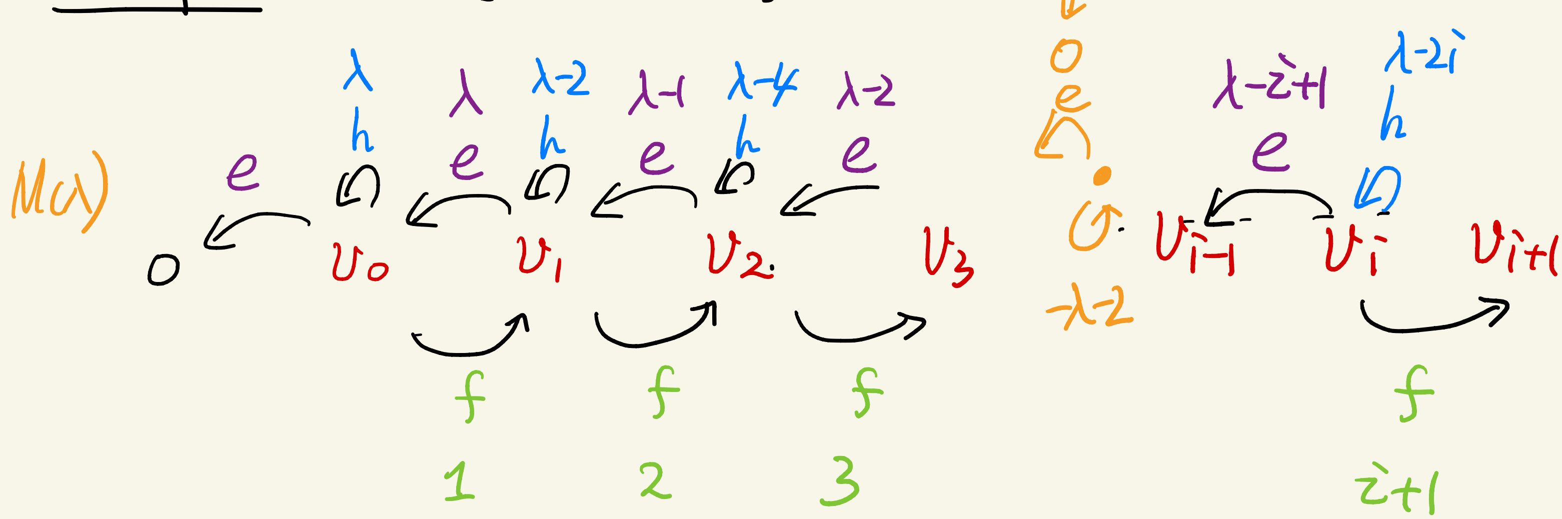
d) $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)^\vee) = 0$

e) $L(\lambda) \cong L(\lambda)^\vee$

f) $\mathrm{Ext}^1_{\mathcal{O}}(M, N) \cong \mathrm{Ext}^1_{\mathcal{O}}(N^\vee, M^\vee)$

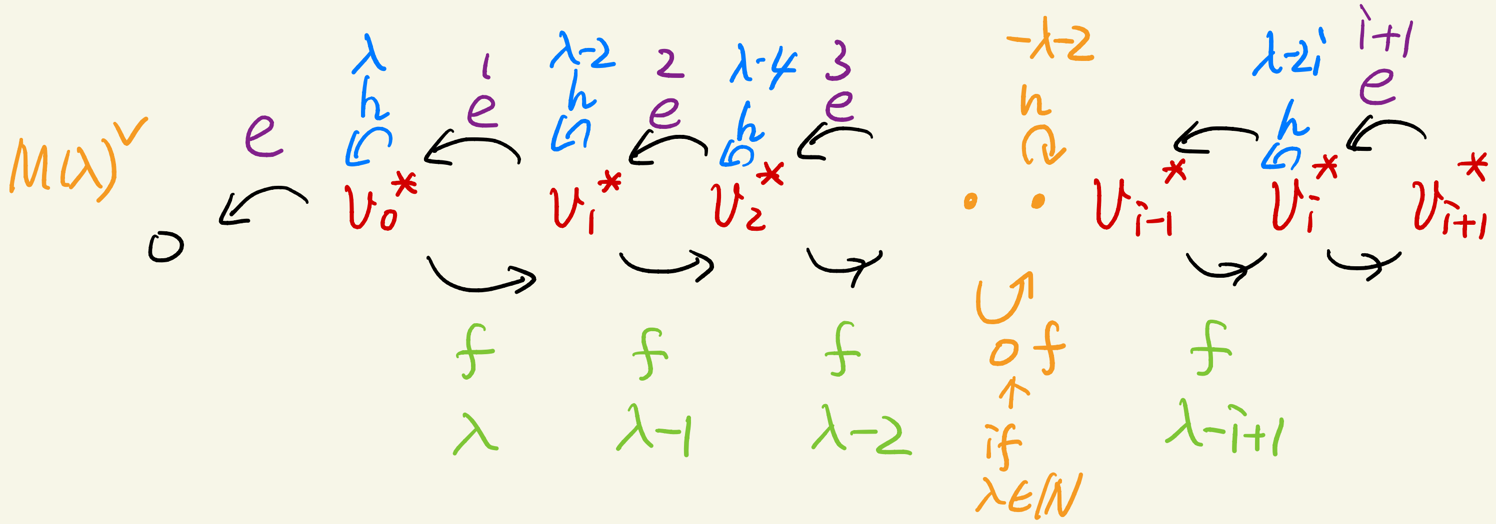
Example

$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ $\quad \lambda \leftrightarrow \lambda\rho \quad$ if $\lambda \in \mathbb{N}$



$M(\lambda)$ is simple $\iff \lambda \notin \mathbb{N}$, $\lambda \in \mathbb{N} \iff \dim L(\lambda) < \infty$

$\lambda \in \mathbb{N} \Rightarrow \quad 0 \to M(-\lambda-2) \to M(\lambda) \to L(\lambda) \to 0$



$\lambda \in \mathbb{N} \Rightarrow \quad 0 \to L(\lambda) \to M(\lambda)^\vee \to M(-\lambda-2)^\vee \to 0$

$M(\lambda)^\vee$ is not a h.wt m if $\lambda \in \mathbb{N}$

Exercise

1) $M(\lambda) \otimes M(\mu) \notin \mathcal{O}$ (not f.g over $U(\mathfrak{n}^-)$)

2) Check $M(\lambda)^\vee \cong M(\lambda)$ $\quad \lambda \in \mathbb{Z}_{\leq -1}$

Blocks of category $\mathcal{O}$

<u>Def</u> $\lambda \in \mathfrak{h}^*$ is

1) antidominant if $\langle \lambda+\rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ $\forall \alpha \in \Phi^+$

2) dominant if $\langle \lambda+\rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$ $\forall \alpha \in \Phi^+$

3) regular if $\langle \lambda+\rho, \alpha^\vee \rangle \neq 0$ $\forall \alpha \in \Phi$ i.e. $|W.\lambda| = |W|$

Define $W_{[\lambda]} := \{ w \in W \mid w\lambda - \lambda \in \mathbb{Z}\Phi \}$

$\lambda \in \Lambda$ (i.e., integral) $\Rightarrow W_{[\lambda]} = W$

$\lambda \in \mathfrak{h}^*$ is antidominant $\Leftrightarrow$ $\lambda \leq w.\lambda$ $\forall w \in W_{[\lambda]}$

There is a unique antidominant weight in $W_{[\lambda]}.\lambda$

<u>Theorem</u> Category $\mathcal{O}$ decomposes as

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_{\chi_\lambda}$$

where $\mathcal{O}_\chi \subset \mathcal{O}$ full subcat consists of M s.t $M = M^\chi$

$M^\chi := \{ v \in M \mid \forall z \in Z(\mathfrak{g})\ \exists n \text{ s.t } (z-\chi(z))^n v = 0 \}$.

i.e., $Z(\mathfrak{g})$ acts via generalised char χ.

Rmk

1) If λ is integral, $\mathcal{O}_{\chi_\lambda}$ is a block of $\mathcal{O}$

2) If $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ $\forall \alpha$ ($w \cdot \lambda$ both dominant and antidominant $\forall w \in W$), then $\mathcal{O}_{\chi_\lambda}$ semisimple

$(P(\mu) = L(\mu) = M(\mu) \quad \forall \mu \in W.\lambda)$

3) $\mathcal{O}_{\chi_{-\rho}} \simeq \{ \text{f.d vector spaces} \}$ $\quad w.(-\rho) = -\rho \ \forall w \in W$

4) Blocks of $\mathcal{O} \xleftrightarrow{1-1} \{\text{antidominant wts}\}$

$\mathcal{O}_\lambda \longleftrightarrow \lambda$ antidominant

$\mathcal{O}_\lambda$: subcategories consisting of modules whose comp. factors all have weights linked to λ (antidominant) by $W_{[\lambda]}$.

$$\mathcal{O}_{\chi_\lambda} = \bigoplus_{\substack{\lambda' \in W.\lambda \\ \lambda' \text{ antidominant}}} \mathcal{O}_{\lambda'}$$

More properties of $\mathcal{O}$

9) $\mathcal{O}$ is Artinian, i.e. every $M \in \mathcal{O}$ is Artinian $\Rightarrow$ Each $M \in \mathcal{O}$ has composition series

Idea of proof

Induction on $\dim U(n) V$ $\quad V = \text{span}\{v_1, \cdots v_n\}$ v_i generating wt vectors

$\Rightarrow \forall M \in \mathcal{O}$ $\quad \exists$ finite filtration

$0 < M_1 < M_2 < \cdots < M_n = M$ s.t M_i / M_{i-1} is h.w.m

$\Rightarrow$ enough to check $M(\lambda)$ has finite length

Thm $\Rightarrow$ The only possible irreducible subquotients are $L(w.\lambda)$ $\quad \dim M(\lambda)_{w.\lambda} < \infty$ $\quad \square$