Riemann-Hilbert Correspondence

<u>References:</u>

- Gelfand - Manin : Homological algebras
- Arkhipov : Lecture notes D-modules.

I) Hilbert's 21 problem:

For $\forall \rho : \pi_1(\mathbb{CP}^1 - \{t_1, \cdots t_N\}, *) \longrightarrow GL(n)$,

Is $\rho =$ monodromy reps of

$n \times n$ matrix

$$\frac{d\,z(t)}{dt} = A(t)\, z(t)$$

a system of n 1^{st} order ODEs on $\mathbb{P}^1_t$.

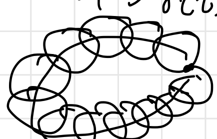
$A(t)\,dt$ has only simple poles contained in $\{t_1, \cdots t_N\}$.

<u>Monodromy:</u>

<u>Thm of ODE:</u> $\mathcal{U}$ simply conn. $\exists$ a unique fundamental solution matrix

$\pi_1 \ni \gamma(t) \quad 0 \leq t \leq 1$

$$S : \mathcal{U} \longrightarrow \underset{GL_n(\mathbb{C})}{\overset{\subseteq}{M_{n\times n}(\mathbb{C})}}$$



$$\gamma(t) \longmapsto S(\gamma(0))^{-1} \cdot S(\gamma(1))$$

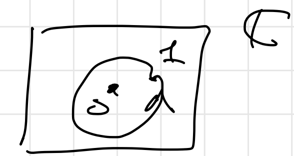
Example: $n=1$

$$\frac{df(z)}{dz} = \frac{a}{z} f(z)$$

ODE on $\mathbb{P}^1 \setminus \{0, \infty\}$

(reg. sing at. $0, \infty$ }

local solution: $f(z) = c z^a$



If $a \notin \mathbb{Z}$, z^a multivalued function.

Monodromy: $\pi_1(\mathbb{P}^1 \setminus \{0, \infty\}) \longrightarrow GL(1) = \mathbb{C}^*$

$\cong \mathbb{Z}$

$z = r e^{i\theta}$

$1 \longmapsto e^{2\pi i a}$ = difference of r^a & $(r^a e^{i 2\pi a})$

Generalization:

- $\mathbb{CP}^1 \rightsquigarrow$ Complex mfd X
- ODE $\rightsquigarrow$
  - × regular singular connection ($\mathcal{O}_X$- coherent D_X-mod) (Deligne 1970)
  - × regular holonomy D-mod. (Kashiwara 1984, Mebkhout 1984)

II) <u>Deligne 1970</u>:

X alg complex mfd.

Def. A connection
- a locally free sheaf of $\mathcal{O}_X$-module $\mathcal{F}$.
- $\mathbb{C}$-linear $\nabla : \mathcal{F} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}$ s.t

$$\nabla(\varphi f) = d\varphi \otimes f + \varphi \otimes \nabla f \qquad \varphi \in \mathcal{O}_X(U),\ f \in \mathcal{F}(U).$$

From ∇, one defines:

$$\nabla^{(p)} : \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \mathcal{F}$$
$$\omega \otimes f \longmapsto d\omega \otimes f + (-1)^p \omega \wedge \nabla^{(0)} f$$

∇ is integrable: if $\nabla^{(p+1)} \circ \nabla^{(p)} = 0$ for all p.

Why $\mathcal{D}_X$-mod ?

Recall: D_X is gen. by $\mathcal{O}_X$, & L_ξ, ξ vector field $\in \Omega_X^\vee$.

$[f, g] = 0, \qquad [L_\xi, f] = \xi(f). \qquad [L_\xi, L_\eta] = L_{[\xi, \eta]}.$

Given an $\mathcal{O}_X$-mod $\mathcal{F}$, with $\nabla : \Omega_X^{1\vee} \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$

L_ξ actions via ∇_ξ. $\qquad \xi \longmapsto \nabla_\xi$

Prop: A D_X-mod is $\mathcal{O}_X$-coherent $\Leftrightarrow$ It's locally free of f. fi rank.

X alg complex mfd. $\qquad X^{an}$ analytic mfd.

$p : (X^{an}, \mathcal{O}^{an}) \to (X, \mathcal{O})$ map of ringed spaces

$\mathcal{F} \in \mathrm{QCoh}(X)$. $\qquad \mathcal{F}^{an} := p^* \mathcal{F}$ on X^{an}.

Thm A: X^{an} conn. analytic nonsingular mfd. Then, the following cats are equivalent

$$\begin{array}{ccc}
(\nabla, \mathcal{F}) & & \mathrm{Conn}(X^{an}) = \{ \text{Integrable connections on } X^{an} \} \\
\downarrow & & \updownarrow \\
\mathcal{E} \text{ the sheaf of flat sections} & & LC(X^{an}) = \{ \text{locally constant sheaves of f. dim v. sp.} \} \\
\mathcal{E}(U) = \{ f \in \mathcal{F}(U) \mid \nabla f = 0 \} & & \updownarrow \\
\downarrow & & \pi_1(X^{an})\text{-mod} = \{ \text{f. dim reps of } X^{an} \} \\
\text{Monodromy reps of } \pi_1 & & \\
\curvearrowright \mathcal{E}_x, \ x \in X & &
\end{array}$$

If X is projective (GAGA)

$$(\nabla, \mathcal{F}) \longrightarrow (\nabla^{an}, \mathcal{F}^{an})$$

<u>Thm A'</u>: X proj. alg var.

$$\mathrm{Conn}(X) \xleftrightarrow{\text{equiv.}}$$

$\mathrm{Conn}(X) = \{\text{integrable conn on } X\}$

- $\mathrm{Conn}(X^{an})$
- $LC(X^{an})$
- $\pi_1(X^{an})$-mod.

<u>Example</u>: For <u>non-cpt</u> mfd. thm A' is wrong.

$X = \mathbb{P}^1 - \{0\}$.

$X^{an} = \mathbb{C} \Rightarrow \pi_1(X^{an}) = \{e\}$

For $\forall P$ polynomial.

$$\begin{array}{c} X \times \mathbb{C} \\ \downarrow \uparrow \varphi(z) \\ X \end{array} \quad \nabla_P = d + \frac{P(\frac{1}{z})}{z^2} dz.$$

$$\Leftrightarrow \frac{d\varphi}{dz} = -\frac{P(\frac{1}{z})}{z^2} \varphi(z).$$

$\longrightarrow$ trivial monodromy.

<u>Flat section</u> $\varphi_P(z) = c\exp\left(-\int^z \frac{1}{\xi^2} P(\tfrac{1}{\xi}) d\xi\right)$

non-equiv singularities at $z=0$.

Thm B (Deligne 1970)

X smooth conn alg mfd $\qquad X^{an}$ analytic mfd.

Let $\mathrm{Conn}_r(X)$ = the cat. of regular integ. connections on X

$= \{ (\mathcal{F}, \nabla) \}$ conn. with regular sing.

locally free $\mathcal{O}_X$-mod

Then: $\mathrm{Conn}_r(X) \cong$ · $\mathrm{Conn}(X^{an})$

· $LC(X^{an})$

· $\pi_1(X^{an})$-mod.

Def: a) If $X = C$ smooth curve. $\quad C \overset{i}{\hookrightarrow} \overline{C}$ complete.
dense

$(\mathcal{F}, \nabla)$ is regular sing. at C if $\forall p \in \overline{C} \setminus C$. $\quad$ locally $p = \{z=0\}$

$\downarrow$

C

$\nabla = d + A(z)$ ← order of poles at p at most 1
$\{z=0\}$

b) For general X. ∇ is regular sing. if $\forall j : C \hookrightarrow X$. $j^*\nabla$ has regular singular. at C.

III). • Generalization of $LC(X^{an})$.

$Sh(X^{an})$ = the cat. of sheaves of vector spaces on X^{an}

$D^b Sh(X^{an})$ = derived cat of $Sh(X^{an})$

$\cup|$

$D^b_{const}(X^{an}) = \{ \mathcal{F}^\cdot : \quad \to \mathcal{F}^{-1} \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \to \cdots \quad H^i(\mathcal{F}) \text{ is constructible.}$

$X^{an} = \bigcup_\alpha S_\alpha \qquad i^*_\alpha(H^i \mathcal{F}) \text{ is locally const of f. r. rank} \}$

$i_\alpha : S_\alpha \hookrightarrow X^{an}$.

• Generalization of $Conn_r(X)$

$D^b(D_X\text{-mod})$

$\cup|$

$D^b_{rh}(D_X\text{-mod})$ ← M is holonomy if $M = \{0\}$, or $ch(M) \subset T^*X$ is Langrangian.

regular singularity

The de Rham & The solution Functor

$$DR\,,\quad Sol \;:\; D^b(D_X\text{-mod}) \longrightarrow D^b(Sh(X^{an}))\,.$$

DR covariant, Sol contravariant

$$DR: \quad M \longmapsto \Omega^{\bullet}_{X^{an}} \otimes_{\mathcal{O}_X^{an}} M^{an}$$

$$= M^{an} \to \Omega^1_X \otimes M^{an} \to \Omega^2_X \otimes M^{an} \to \cdots$$

$$Sol: \quad M \longmapsto R\mathcal{H}om_{D_X^{an}}(M, \mathcal{O}_X^{an})\,.$$

Why called solution functor?

PDEs $\rightsquigarrow$ $M := D_X \Big/ \sum_{i=1}^{m} D_X P_i$

$P_1, \cdots, P_m$

diff. operators

The space of solutions

$$\mathrm{Hom}(M, \mathcal{O}_X^{an})$$

$M = D_X/I$

$$1 \longmapsto \varphi(1)$$

$$\boxed{P_i(\varphi(1)) = 0\,.}$$

Prop.:

$$\boxed{\mathrm{Sol}(M) = DR(\mathbb{D}(M))\,[-\dim X].}$$

- $DR(M) = \Omega_X^{\bullet\, an} \otimes_{\mathcal{O}^{an}} M^{an}$ $\longleftarrow$ $H^i(\Omega_X^{an} \otimes_{\mathcal{O}_X} \mathcal{D}^{an}) = \begin{cases} \Omega^{top} & i = \dim X \\ 0 & \end{cases}$

$$= \left(\Omega^{an}_{top} \overset{L}{\otimes}_{\mathcal{D}^{an}} M^{an}\right)[-\dim X]$$

- $\mathrm{Sol}(M) = \underline{R\mathcal{H}om}_{\mathcal{D}_X^{an}}(M, \mathcal{O}_X^{an})$

$$= \mathbb{D}(M) \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{O}_X^{an}.$$

- Exercise: Take $M = \mathcal{D}_X \otimes V$

  check the relation

$\blacksquare$